203 §14.3 #5 4, 10, 15–20, 33–38, 48, 51, 53, 77

(4) h = height of wave as function of v = velocity of wind and t = duration
knots hours

$h(v, t)$

(a) $\frac{\partial h}{\partial v}$ = change in height over a small increment of velocity.

$\frac{\partial h}{\partial t}$ = change in height over small increment of time.

(b) We estimate $f_v(40,15)$ and $f_t(40,15)$

$f(40,15) = 25$

$f(50,15) = 36$

$f(30,15) = 16$

$$\frac{f(50,15) - f(40,15)}{50-40} = \frac{36-25}{10} = \frac{11}{10} = 1.1$$

$$\frac{f(40,15) - f(30,15)}{40-30} = \frac{25-16}{10} = \frac{9}{10}$$

AVG $\approx \boxed{1 \text{ ft/knot}}$

$$f_v(40,15) \approx 1 \frac{\text{ft}}{\text{knot}}$$

203 §14.3 #s 15-20, 33-38, 48, 51, 53, 77

#s 15-40 Find 1st partials

15 $f(x,y) = y^5 - 3xy$

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$f_x = -\pi e^{-t}\sin(\pi x)$

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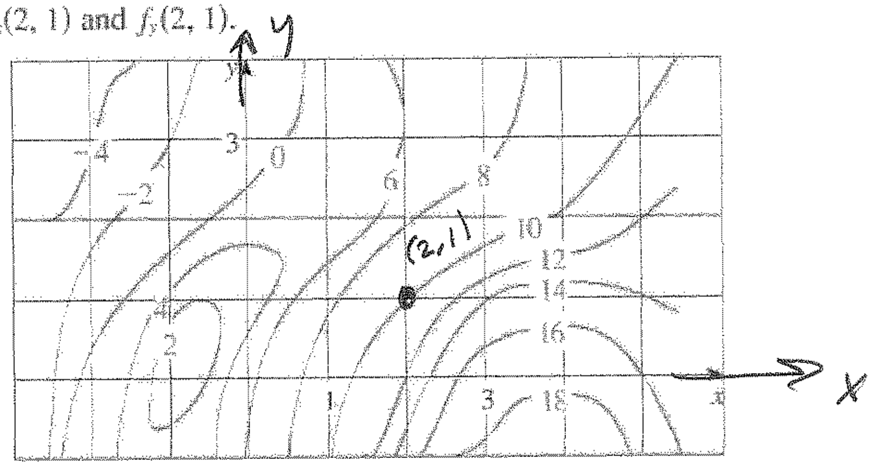
$f_x = \frac{1}{2}x^{-\frac{1}{2}}\ln t$

$f_t = \frac{\sqrt{x}}{t}$

203 §14.3 #s 10, 15-20, 33-38, 48, 51, 53, 77

10. A contour map is given for a function f. Use it to estimate $f_x(2, 1)$ and $f_y(2, 1)$.



$f_x(2,1)$: $f(2,1) = 10$

$f(3,1) = 14$

$f(1,1) \approx 8$

$\frac{f(2,1) - f(3,1)}{2-3} = \frac{10-14}{-1} = 4$

$\frac{f(2,1) - f(1,1)}{2-1} \approx \frac{10-8}{1} = 2$

AVG : $\boxed{f_x(2,1) \approx 3.}$

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$f(2,2) \approx 8$

$f(2,0) = 12$

$\frac{f(2,1) - f(2,2)}{1-2} = \frac{10-8}{-1} = -2$

$\frac{f(2,1) - f(2,0)}{1-0} = \frac{10-12}{1} = -2$

$\boxed{f_y(2,1) \approx -2}$

203 §14.3 #s 4, 10, 15-20, 33-38, 48, 51, 53, 77

(4) (b) cont'd: $f_t(40,15)$: v is fixed @ v = 40.

$f(40,15) = 25$, $f(40,20) = 28$

$f(40,10) = 21$

$$\frac{f(40,20) - f(40,15)}{20-15} = \frac{28-25}{5} = \frac{3}{5}\ \frac{ft}{hr}$$

$$\frac{f(40,15) - f(40,10)}{15-10} = \frac{25-21}{5} = \frac{4}{5}\ \frac{ft}{hr}$$

AVG $= \frac{3+4}{10} = \frac{7}{10}$

$\boxed{f_t(40,15) \approx \frac{7}{10}\ \frac{ft}{hr}}$ ≈ rate at which wave heights are rising wrt time, with wind velocity held constant @ 40 knots.

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203 § 14.3 #s 19, 20, 33–38, 48, 51, 53, 77

(19) $z = (2x+3y)^{10}$

$z_x = 10(2x+3y)^9(2)$

$z_y = 10(2x+3y)^9(3)$

(20) $z = \tan(xy)$

$z_x = y\sec^2(xy)$

$z_y = x\sec^2(xy)$

(33) $w = \ln(x+2y+3z)$

$w_x = \dfrac{1}{x+3y+3z}$, $w_y = \dfrac{2}{x+2y+3z}$, $w_z = \dfrac{3}{x+2y+3z}$

(34) $w = ze^{xyz}$

$w_x = yz^2e^{xyz}$, $w_y = xz^2e^{xyz}$, $w_z = xyze^{xyz}$

(35) $u = xy\sin^{-1}(yz)$

$u_x = y\sin^{-1}(yz) + \cancel{\left(xy\sqrt{1-(yz)^2}\right)(0)}$

$u_y = x\sin^{-1}(yz) + \left(xy\left(\dfrac{1}{\sqrt{1-(yz)^2}}\right)z\right.$

$u_z = \left(xy\left(\dfrac{1}{\sqrt{1-(yz)^2}}\right)y\right.$

$\dfrac{d}{dx}\left[f^{-1}(x)\right] = \dfrac{1}{f'(f^{-1}(x))}$

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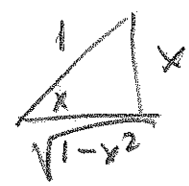

203 §14.3 #s 36-38, 48, 51, 53, 77

(36) $u = x^{y/z} = x^{yz^{-1}}$

$$\boxed{\begin{aligned} u_x &= (yz^{-1})x^{yz^{-1}-1} \\ u_y &= \left(\frac{1}{z}\ln x\right)x^{\frac{y}{z}} \end{aligned}}$$

$\frac{d}{dx} \ln u = \left(\frac{y}{z}\right)\ln x$

$\frac{u'}{u} = \frac{y}{zx}$

$\frac{\partial u}{\partial x} = \frac{y}{zx} x^{yz^{-1}}$

$= \frac{y}{z} x^{yz^{-1}-1}$

$\frac{d}{dy}$: $x^{\frac{y}{z}} = e^{\ln(x^{\frac{y}{z}})}$

$= e^{\frac{y}{z}\ln x}$

$= e^{\left(\frac{\ln x}{z}\right)y} \longrightarrow$

$\frac{d}{dy}(x^{y/z}) = \frac{\ln x}{z} e^{\frac{\ln x}{z} y}$

$= \left(\frac{1}{z}\ln x\right) e^{\frac{y}{z}\ln x}$

$= \left(\frac{1}{z}\ln x\right) e^{\ln(x^{\frac{y}{z}})}$

$= \left(\frac{1}{z}\ln x\right) x^{\frac{y}{z}}$

I can never remember

$\frac{d}{dx}[b^x] = (\ln(b))b^x$

So I re-derive it, any time it's been a while...

203 §14.3 #s 37-8, 48, 51, 53, 77

(37) $h(x, y, z, t) = x^2 y \cos\left(\frac{z}{t}\right)$

$$h_x = 2xy \cos\left(\frac{z}{t}\right)$$

$$h_y = x^2 \cos\left(\frac{z}{t}\right)$$

$$h_z = x^2 y \left(-\sin\left(\frac{z}{t}\right)\right)\left(\frac{1}{t}\right) = -\frac{x^2 y}{t} \sin\left(\frac{z}{t}\right)$$

$$h_t = x^2 y \left(-\sin\left(\frac{z}{t}\right)\right)\left(-\frac{z}{t^2}\right) = \frac{x^2 y z}{t^2} \sin\left(\frac{z}{t}\right)$$

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$$\phi_x = \frac{\partial}{\partial x}\left[\frac{1}{\gamma z + \delta t^2}\left(\alpha x + \beta y^2\right)\right] = \boxed{\frac{\alpha}{\gamma z + \delta t^2}}$$

$$\phi_y = \frac{\beta}{\gamma z + \delta t^2}$$

$$\phi_z = \frac{\partial}{\partial z}\left[(\alpha x + \beta y^2)(\gamma z + \delta t^2)^{-1}\right]$$

$$= (-1)(\gamma z + \delta t^2)^{-2}(\gamma)(\alpha x + \beta y^2)$$

$$= -\frac{\gamma(\alpha x + \beta y^2)}{(\gamma z + \delta t^2)^2}$$

$$\phi_t = (-1)(\gamma z + \delta t^2)^{-2}(2\delta t)(\alpha x + \beta y^2)$$

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203 §14.3 #s 48, 51, 53, 77

~~48~~ #s 47-50 use implicit diff to find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$

48 $x^2 - y^2 + z^2 - 2z = 4$

See Formula 7, pg 954, §14.5. I usually don't bother with it, but you might like using it.

Easier for me to just use the old skills. Technically, we don't "know" formula 7, yet.

$z_x = \frac{\partial z}{\partial x}$:

$2x + 2z z_x - 2z_x = 0$

$(2z-2) z_x = -2x$

$$\boxed{z_x = \frac{-2x}{2z-2}}$$

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203 §14.3 #s 51, 53, 77

#s 51-2 Find z_x & z_y

51 (a) $z = f(x) + g(y)$

$z_x = f'(x)$, $z_y = g'(y)$

(b) $z = f(x+y) \longrightarrow$

$z_x = f_x(x+y)$

$z_y = f_y(x+y)$

53 Find all 2nd partials

$f(x,y) = x^3y^5 + 2x^4y$

$f_x = 3x^2y^5 + 8x^3y$ $\qquad f_y = 5x^3y^4 + 2x^4$

$\qquad f_{yy} = 20x^3y^3$

$f_{xx} = 6xy^5 + 24x^2y$

$f_{xy} = 15x^2y^4 + 8x^3 = f_{yx} = 15x^2y^4 + 8x^3$ ✓

77 Verify: $u = (x^2+y^2+z^2)^{-\frac{1}{2}}$ is soln of Laplace's eq'n in 3-D: $u_{xx} + u_{yy} + u_{zz} = 0$

$u_x = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2x)$

$u_y = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2y)$

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203 §14.3 #s 4, 10, 15–20, 33–38, 48, 51, 53, 77

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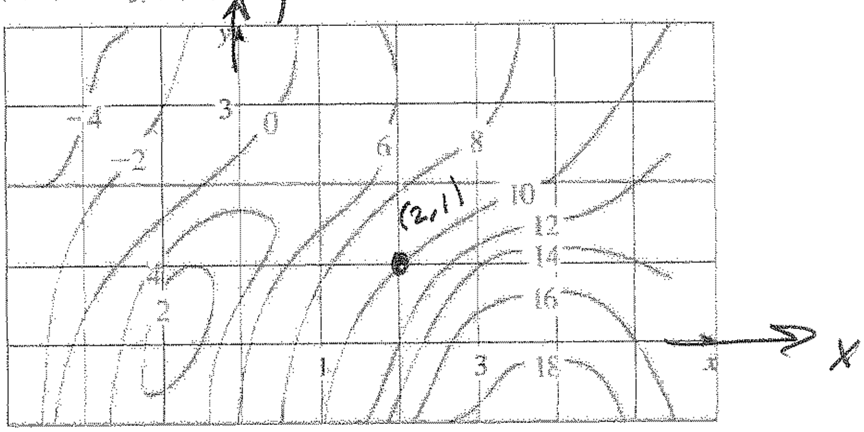

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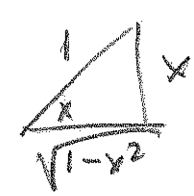

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203 § 14.3 #77

$$u_{xx} = \frac{3}{4}(x^2+y^2+z^2)^{-\frac{5}{2}}(2x)(2x) - \frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2)$$

$$u_{yy} = \frac{3}{4}(x^2+y^2+z^2)^{-\frac{5}{2}}(2y)(2y) - \frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2)$$

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Times $\frac{x^2+y^2+z^2}{x^2+y^2+z^2}$

$\longrightarrow u_{xx} + u_{yy} + u_{zz}$

$$= \frac{3x^2+3y^2+3z^2}{(x^2+y^2+z^2)^{5/2}} - \frac{3}{(x^3+y^2+z^2)^{3/2}}$$

$$= 3\left[\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}} - \frac{(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}}\right] = 0$$

$\blacksquare$