§16.9

recall vector-valued functions, defined parametrically

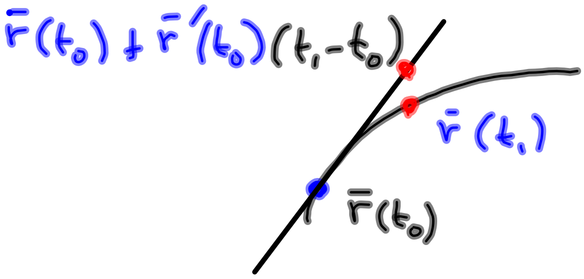
$$\bar{r}(t) = \langle x(t), y(t) \rangle$$

$$\bar{r}'(t) = \langle x'(t), y'(t) \rangle$$

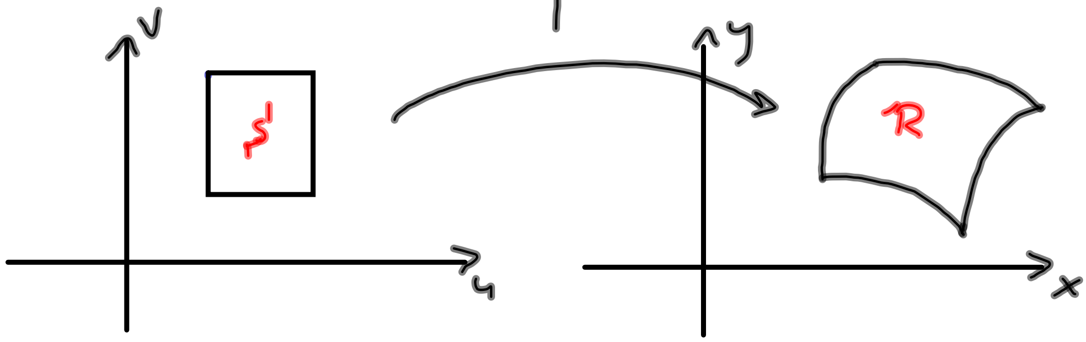
Linear approximation

$$\bar{r}(t_1) \approx \bar{r}(t_0) + \bar{r}'(t_0)(t_1 - t_0)$$

$$= \bar{r}(t_0) + \bar{r}'(t_0)\Delta t$$



$$\iint_S f(u,v)\,dA \xrightarrow{\text{want}} \iint_R f(x(u,v),y(u,v))\,\text{"}dA\text{"}$$



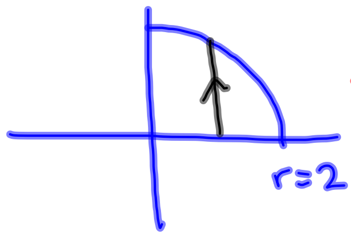
$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$T((u,v)) = (x,y) = (x(u,v),\ y(u,v))$$

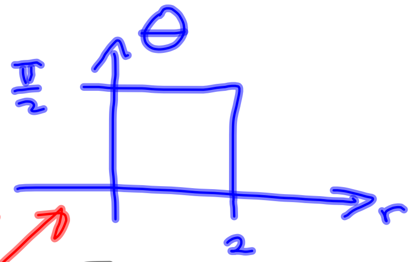
We've done stuff like this, already.

$$\iint (x^2+y^2)\,dy\,dx \longrightarrow \iint r^2 \cdot r\,dr\,d\theta$$

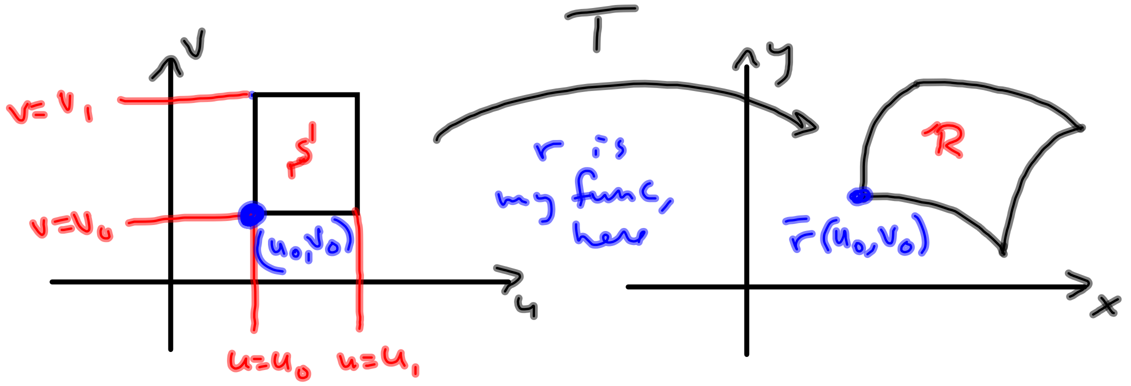
From rectangular to polar.



Turns the quarter-circle in the xy-plane onto a rectangle



$$\int_0^2 \int_0^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx \longrightarrow \int_0^{\frac{\pi}{2}} \int_0^2 r^2 \cdot r\,dr\,d\theta$$

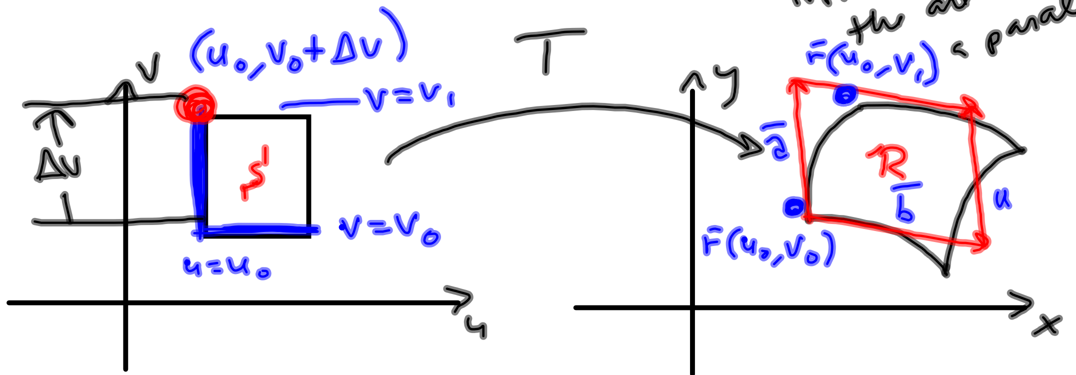


$\Delta u = u_1 - u_0$

$\Delta v = v_1 - v_0$

$\bar{r}(u,v) = \langle x(u,v), y(u,v) \rangle$

Approximate the area with a parallelogram



Area of the parallelogram is $|\bar{a}\times\bar{b}|$

We "build" $\bar{a}$ using linear approximation

See the two points $\bar{r}(u_0,v_0)$ & $\bar{r}(u_0,v_1)$

$\bar{r}(u_0,v_1) = \bar{r}(u_0, v_0+\Delta v)$

$$\Delta r = \bar{r}(u_0, v_0+\Delta v) - \bar{r}(u_0,v_0)$$

$$\frac{\Delta r}{\Delta v} = \frac{\bar{r}(u_0,v_0+\Delta v)-\bar{r}(u_0,v_0)}{\Delta v} \approx \frac{\partial \bar{r}}{\partial v}$$

So, $\bar{a} \approx \bar{r}(u_0, v_0+\Delta v) - \bar{r}(u_0, v_0)$

$$\approx \frac{\partial \bar{r}}{\partial v} \cdot \Delta v = \bar{r}_v \Delta v$$

Likewise, $\bar{b} \approx \bar{r}_u \Delta u$

And so, $|\bar{a} \times \bar{b}| \approx |(\bar{r}_v \Delta v) \times (\bar{r}_u \Delta u)|$

$= |\bar{r}_v \times \bar{r}_u| \Delta v \Delta u$

$\approx$ Area of the region $\mathcal{R}$, using the mapping T & the domain S

and so

$$\iint_S f(u,v)\, dA$$

Change of variables for double integrals.

$$= \iint_{\mathcal{R}} f(x(u,v), y(u,v))\, \underbrace{|\bar{r}_u \times \bar{r}_v|}\, du\, dv$$

$$\bar{r}_u \times \bar{r}_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{K} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{vmatrix} = (x_u y_v - x_v y_u)\bar{K}$$

$\bar{r}_u = \langle x_u, y_u \rangle$

$$\text{Jacobian} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{K} \\ x_u & x_v & 0 \\ y_u & y_v & 0 \end{vmatrix} = (x_u y_v - x_v y_u)\bar{K}$$

And we just want its absolute value, not its direction.

$\boxed{D}$ Jacobian $= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$

D7

$$= \frac{\partial(x,y)}{\partial(u,v)}$$

$$\Delta A = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v$$

Convert

$$\int_0^1 \int_0^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx = \int_0^{\frac{\pi}{2}} \int_0^2 r^2 \cdot r\,dr\,d\theta$$

$$\bar{r} = \langle x(r,\theta), y(r,\theta) \rangle$$

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \quad \begin{aligned} \bar{r}_r &= \langle \cos\theta, \sin\theta \rangle \\ \bar{r}_\theta &= \langle -r\sin\theta, r\cos\theta \rangle \end{aligned}$$

$$\left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = \left\| \begin{matrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{matrix} \right\| = \left\| \begin{matrix} x_r & x_\theta \\ y_r & y_\theta \end{matrix} \right\|$$

$$= |r\cos^2\theta + r\sin^2\theta|$$

$$= |r(\cos^2\theta + \sin^2\theta)| = |r| = r$$