§16.2 Iterated Integrals

$$\int_0^1 \int_1^2 (4x^3 - 9x^2y^2)\, dy\, dx$$

$$= \int_0^1 \left[4x^3y - 3x^2y^3 \right]_{1=y}^{2=y} dx$$

$$= \int_0^1 \left(8x^3 - 24x^2 - \left[4x^3 - 3x^2 \right] \right) dx$$

$$= \int_0^1 (4x^3 - 21x^2)\, dx = \left[x^4 - 7x^3 \right]_0^1 = 1 - 7 = -6$$

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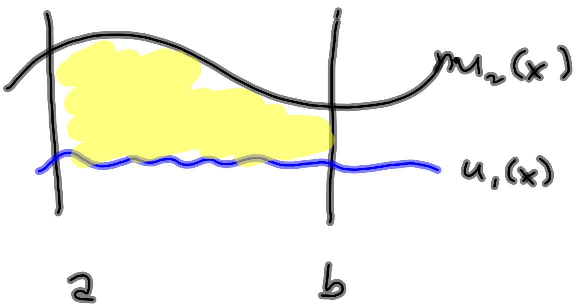
§16.3 Integrating over more general regions

Type I



$z = f(x,y)$ is a surface defined for all $(x,y) \in \mathcal{D}$.

$$\iint_{\mathcal{D}} f(x,y)\, dA = \int_a^b \int_{u_1(x)}^{u_2(x)} f(x,y)\, dy\, dx$$

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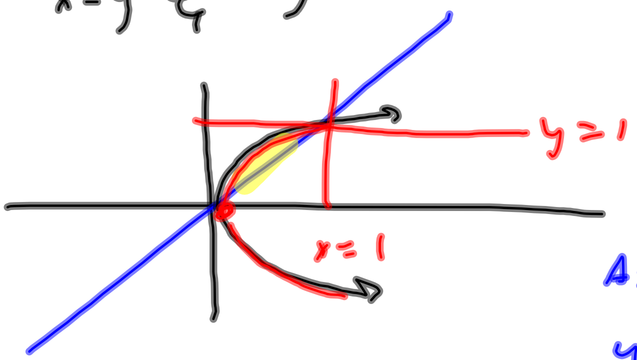
Find volume of region bounded by $z=2x+y^2$, above the region bdd. by $x=y$ & $x=y^2$

$y=y^2 \Rightarrow y=\pm 1$



As a type I region:

$y=x$ is lower $=u_1(x)$

$x=y^2=x$ Not nice for type I approach $y=\sqrt{x}$ ∴ upper $=u_2(x)$

$y=\pm\sqrt{x}$ want $+\sqrt{x}$

$$\int_0^1\int_{y=x}^{y=\sqrt{x}}(2x+y^2)\,dy\,dx$$

$$=\int_0^1\left[2xy+\frac{1}{3}y^3\right]_{y=x}^{y=\sqrt{x}}dx$$

$$=\int_0^1\left[\underline{2x\sqrt{x}}+\frac{1}{3}x^{\frac{3}{2}}-\left(2x^2+\frac{1}{3}x^3\right)\right]dx$$

$$=\int_0^1\left[\frac{7}{3}x^{\frac{3}{2}}-2x^2-\frac{1}{3}x^3\right]dx$$

$$=\left[\frac{7}{3}\cdot\frac{2}{5}x^{\frac{5}{2}}-\frac{2}{3}x^3-\frac{1}{12}x^4\right]_0^1$$

$$=\frac{14}{15}-\frac{2}{3}-\frac{1}{12}=\frac{56-40-5}{60}=\boxed{\frac{11}{60}}$$

Type I kinda sucked,

Type II holds promise!

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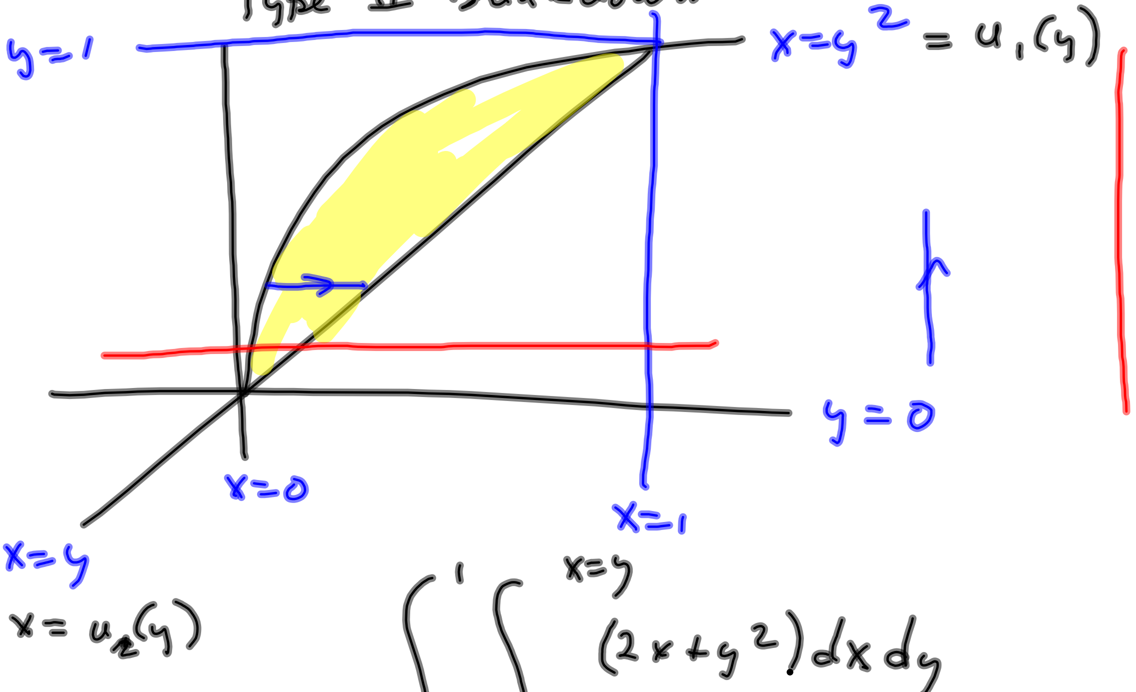
Type II Breakdown:



$$\int_0^1 \int_{x=y^2}^{x=y} (2x+y^2)\,dx\,dy$$

$$= \int_0^1 \left[x^2 + xy^2\right]_{x=y^2}^{x=y} dy$$

$$= \int_0^1 \left[y^2 + y^3 - (y^4 + y^4)\right] dy$$

$$= \int_0^1 \left[y^2 + y^3 - 2y^4\right] dy$$

$$= \left[\frac{1}{3}y^3 + \frac{1}{4}y^4 - \frac{2}{5}y^5\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4} - \frac{2}{5}$$

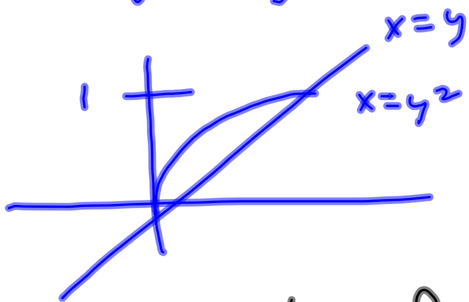
$$= \frac{20+15-24}{60} = \frac{11}{60}$$

$$\int_0^1 \int_{x=y^2}^{x=y} (2x+y^2)\,dx\,dy = \int_0^1 \int_{y=x}^{y=\sqrt{x}} (2x+y^2)\,dy\,dx$$

Later in the chapter, we'll be switching the order of integration

$$\int_0^1 \int_{x=y^2}^{x=y} f(x,y)\,dx\,dy$$

Integrating over a Type II region.



To swap order of integration, we need to convert to a Type I.

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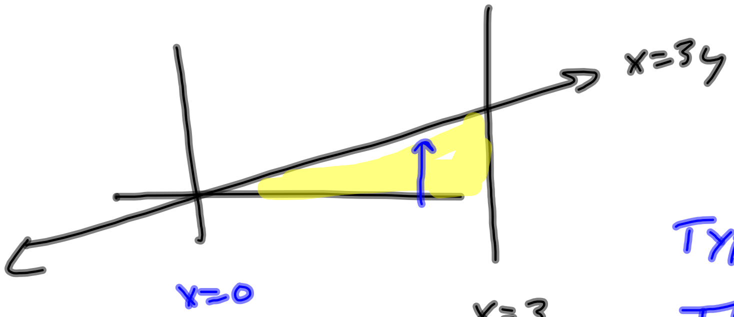
Example 3 is cool, because the Type I requires 2 integrals, because the lower function $g_1(x)$ changes @ $x=-1$.

45 $$\int_0^1 \int_{x=3y}^{x=3} e^{x^2}\,dx\,dy$$

viewed as Type II

e^{x^2} has no closed-form antiderivative.

$y = \frac{1}{3}x$



Type I approach: This gives

Type I: $g_1(x) = 0$
$g_2(x) = \frac{1}{3}x$

$$\int_0^3 \int_{y=0}^{\frac{1}{3}x} e^{x^2}\,dy\,dx$$

$$= \int_0^3 \Big[y e^{x^2} \Big]_{y=0}^{y=\frac{1}{3}x} dx$$

$$= \int_0^3 \Big[\tfrac{1}{3} x e^{x^2} \Big] dx = \frac{1}{3}\cdot\frac{1}{2}\int_0^3 e^{x^2}\cdot 2x\,dx$$

$$= \frac{1}{6}e^{x^2}\Big]_0^3 = \boxed{\frac{1}{6}e^9 - \frac{1}{6}}$$

Sweet!

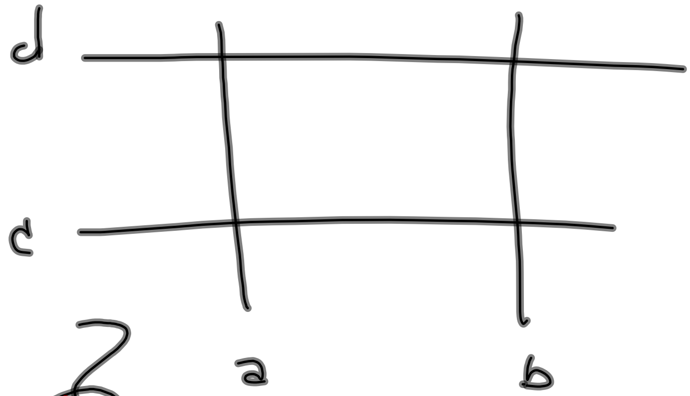
Fubini Said

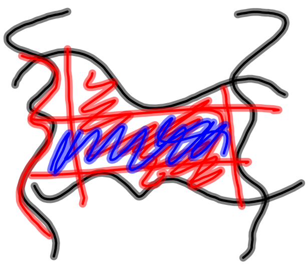
$$\int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Only works over rectangles:

$$[a,b] \times [c,d]$$





$$\frac{d}{dx} \int_a^x \int_b^y g(t,s)\,ds\,dt$$

$$= \int_b^y g(x,s)\,ds$$

$$\frac{d}{dx} \int_a^x f(s)\,ds = f(x)$$