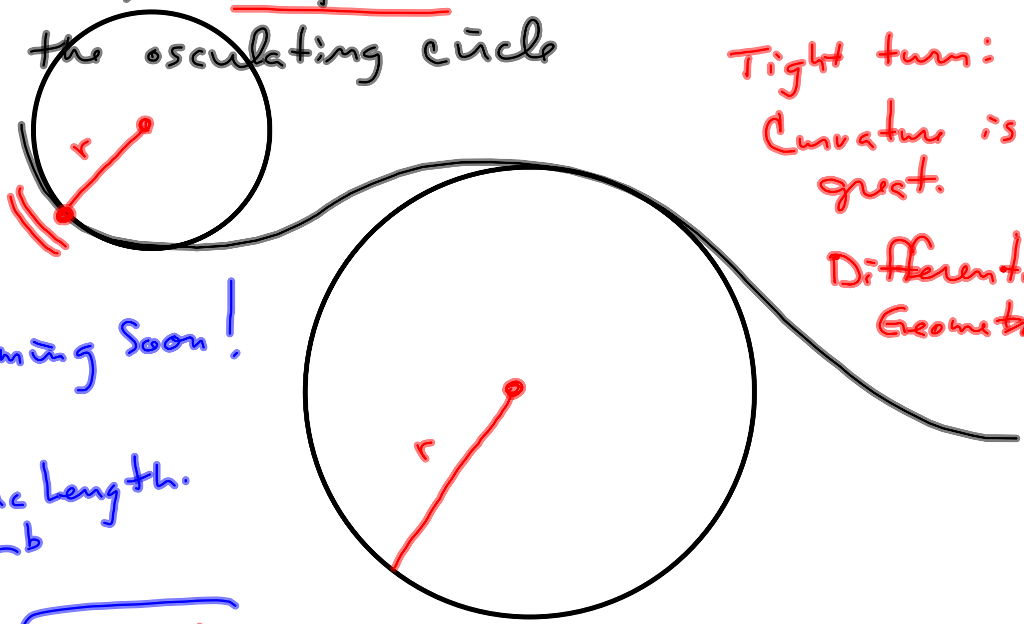
§14.3 Arc Length and Curvature

<u>Curvature</u>

→ The <u>reciprocal</u> of the radius of the osculating circle



Tight turn: Curvature is great.

Differential Geometry.

Coming Soon!

Arc Length.

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt$$

From the olden days.

$x = f(t)$

$y = g(t)$

Now

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

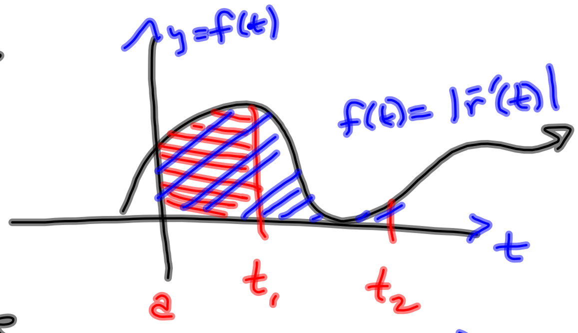
For $\bar{r}(t) = \langle x(t), y(t), z(t) \rangle$

$$S = L = \int_a^b |\bar{r}'(t)|\, dt$$ t=a be starting point.

Arc Length Function:

$$s(t) = \int_a^t |\bar{r}'(u)|\, du$$ is an increasing function of t.

We often parametrize a curve C $(\bar{r}(t))$ in terms of arc length.
"Where am I in space, after I've travelled some distance, s?"



$s(t_2) > s(t_1)$

#14 Re-parametrize w.r.t. s, from $t=0$, t increasing

$\bar{r}(t)=\langle e^{2t}\cos(2t), 2, e^{2t}\sin(2t)\rangle$

Find $s(t)$. Solve for t, giving $t=t(s)$.

$$\bar{r}'(t)=\left(2e^{2t}\cos(2t)+e^{2t}(-2\sin(2t))\right)\bar{i}$$

$$+ \quad 0\,\bar{j}$$

$$+\left(2e^{2t}\sin(2t)+e^{2t}(2\cos(2t))\right)\bar{k}$$

$$= \langle 2e^{2t}(\cos(2t)-\sin(2t)), 0, 2e^{2t}(\sin(2t)+\cos(2t))\rangle = \bar{r}'(t)$$

$$s(t)=\int_0^t |\bar{r}'(u)|\,du$$

$$|\bar{r}'(t)| = \sqrt{4e^{4t}\underline{(\cos(2t)-\sin(2t))}^2+4e^{4t}(\underline{\sin(2t)+\cos(2t)})^2}$$

Scratch:

$\cos^2(2t) - 2\cos(2t)\sin(2t) + \sin^2(2t) + \sin^2(2t) + 2\sin(2t)\cos(2t) + \cos^2(2t)$

$= 2$

$$\therefore |\bar{r}'(t)| = \sqrt{4e^{4t}(\;2\;)} = \sqrt{8e^{4t}} = 2e^{2t}\sqrt{2}$$

$$\therefore \quad s(t) = \int_0^t |\bar{r}'(u)|\,du = \int_0^t 2\sqrt{2}e^{2u}\,du = 2\sqrt{2}\left(\tfrac{1}{2}e^{2u}\right)\Big|_0^t$$

$$= \sqrt{2}\left(e^{2t} - e^{0}\right) = \sqrt{2}e^{2t} - \sqrt{2} = s(t)$$

$$\sqrt{2}e^{2t} - \sqrt{2} = s$$

$$\sqrt{2}e^{2t} = s + \sqrt{2}$$

$$e^{2t} = \frac{s+\sqrt{2}}{\sqrt{2}}$$

$$2t = \ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right)$$

$$t = \frac{1}{2}\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right) = \ln\sqrt{\frac{s+\sqrt{2}}{\sqrt{2}}}$$

$$\bar{r}(t) = \langle e^{2t}\cos(2t),\ 2,\ e^{2t}\sin(2t)\rangle$$

$$\bar{r}(t(s)) = \left\langle e^{2\ln\sqrt{\frac{s+\sqrt{2}}{\sqrt{2}}}}\cos\left(2\ln\sqrt{\frac{s+\sqrt{2}}{\sqrt{2}}}\right),\ 2,\ e^{2\ln\sqrt{\frac{s+\sqrt{2}}{\sqrt{2}}}}\sin\left(2\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right)\right)\right\rangle$$

$$= \bar{r}(s)$$

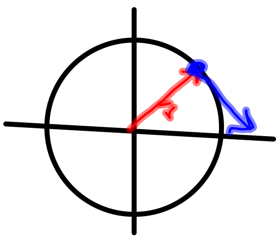
§14.3 II $\quad \bar{r}'(t)$ cnt$^{\underline{s}}$ $\Big\}$ Smooth
$\bar{r}'(t) = \bar{0}$?

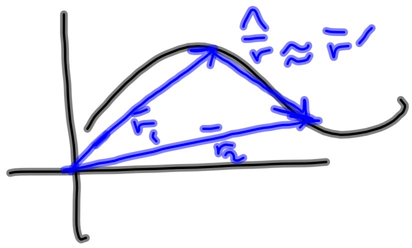
unit tangent for $\bar{r}(t)$ is

$$\bar{T}(t) = \frac{\bar{r}'(t)}{|\bar{r}'(t)|}$$

Recall, if a vector lies on a circle centered @ O, then the tangent to the curve is at right angles to the vector.

Better-worded on Page 863, Ex #4.





Think of $\bar{T}$ as a POSITION vector.

$$|\bar{T}(t)| = 1$$

$\therefore$ $\bar{T}$ is orthogonal to its derivative!

$$\bar{T} \cdot \bar{T}' = 0$$

$\bar{N}(t)$ = unit Normal Vector = $\dfrac{\bar{T}'(t)}{|\bar{T}'(t)|}$

$\bar{B}(t) = \bar{T}(t) \times \bar{N}(t)$ = Binormal Vector.

$\bar{T}$ & $\bar{N}$ lie in the osculating plane. $\bar{B}$ is the (a) normal to this plane.