13.1 THREE-DIMENSIONAL COORDINATE SYSTEMS

13.1 #s 3, 8, 9, 10, 11, 16, 20, 29, 35
13.2 #s 4, 5, 13, 15, 17, 21, 27, 29, 41, 42, 45

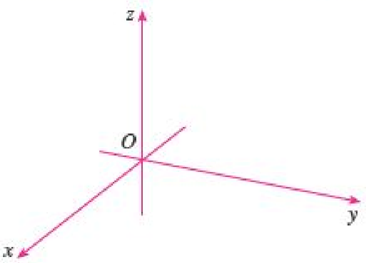
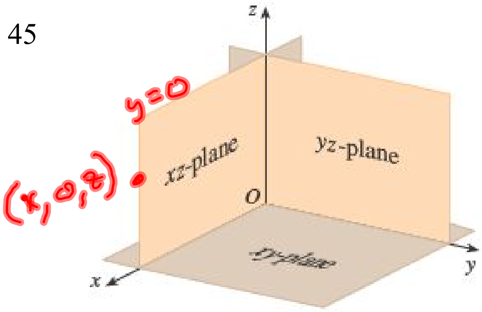
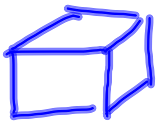


FIGURE 1
Coordinate axes



(a) Coordinate planes



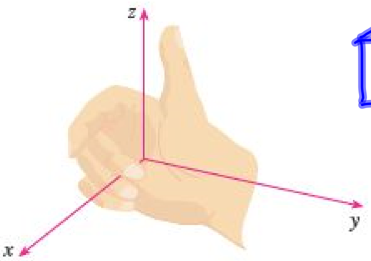
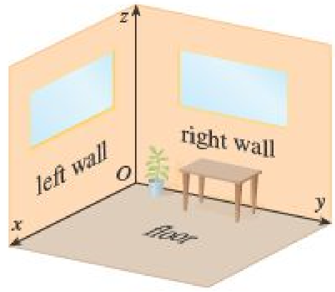


FIGURE 2
Right-hand rule



(b)

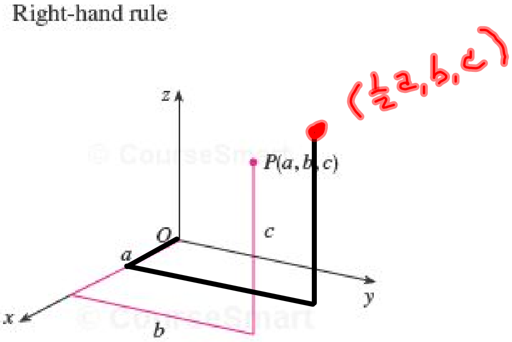


FIGURE 4

$(\frac{1}{2}a, b, c)$

We represent the point P by the ordered triple (a, b, c) of real numbers and we call a, b, and c the **coordinates** of P; a is the x-coordinate, b is the y-coordinate, and c is the z-coordinate.

The point $P(a, b, c)$ determines a rectangular box as in Figure 5. If we drop a perpendicular from P to the xy-plane, we get a point Q with coordinates $(a, b, 0)$ called the **projection** of P on the xy-plane. Similarly, $R(0, b, c)$ and $S(a, 0, c)$ are the projections of P on the yz-plane and xz-plane, respectively.

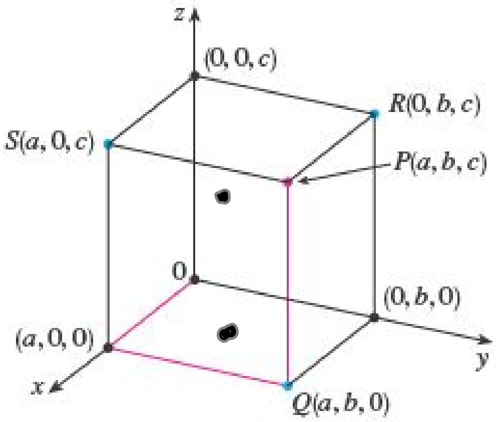
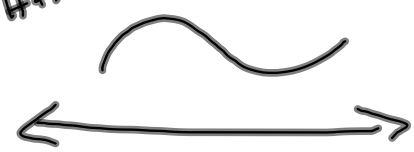


FIGURE 5

$\bar{a} \times \bar{b}$

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Not "Cross Product" → Has specific meaning



The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$. We have given a one-to-one correspondence between points P in space and ordered triples (a, b, c) in $\mathbb{R}^3$. It is called a **three-dimensional rectangular coordinate system**. Notice that, in terms of coordinates, the first octant can be described as the set of points whose coordinates are all positive.

We now speak of surfaces in $\mathbf{R}^3$, whereas we used to speak of curves in $\mathbf{R}^2$.

$y = f(x)$ becomes $z = f(x, y)$

1-D embedded in 2-space

A surface

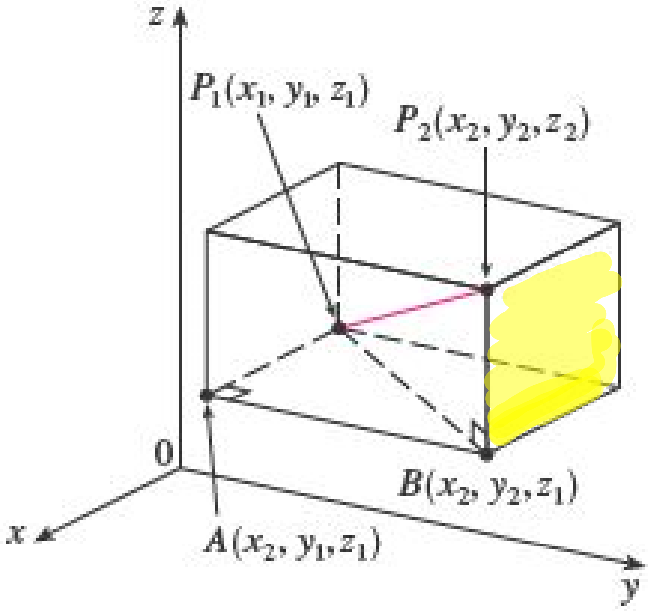
2-D object embedded in 3-space.

3 space is divided into (surprisingly enough) 8 octants.

DISTANCE FORMULA IN THREE DIMENSIONS The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Proof of this is found by application of Pythagorean Theorem in $\mathbf{R}^2$. We're helped by projections onto appropriate planes. To what planes am I referring?



Briefly,

$|\overrightarrow{P_1B}|^2 = d(P_1, B)^2$

$= |P_1A|^2 + |AB|^2$

$= |x_1 - x_2|^2 + |y_1 - y_2|^2$

Etc.

I've shaded the plane $x = x_2$ in black.

FIGURE 9

$|P_1A| = |x_2 - x_1|$ $|AB| = |y_2 - y_1|$ $|BP_2| = |z_2 - z_1|$

$$|P_1P_2|^2 = |P_1B|^2 + |BP_2|^2$$

$$|P_1B|^2 = |P_1A|^2 + |AB|^2$$

$$|P_1P_2|^2 = |P_1A|^2 + |AB|^2 + |BP_2|^2$$
$$= |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

EXAMPLE 4 Find an equation of a sphere with radius r and center $C(h, k, l)$.

Circle in 2-D: $\sqrt{(x-h)^2 + (y-k)^2} = \sqrt{r^2}$

Distance from (x,y) on the circle to the center @ (h,k) is r.

Let (x,y,z) be a point on the sphere.

Then $\underline{(x-h)^2 + (y-k)^2 + (z-l)^2} = r^2$

Sphere

(h,k,l) = center

$\sqrt{(x-h)^2 + (y-k)^2 + (z-l)^2}$ is distance from (x,y,z) to (h,k,l)

EQUATION OF A SPHERE An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

In particular, if the center is the origin O, then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$

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7–8 Find the lengths of the sides of the triangle PQR. Is it a right triangle? Is it an isosceles triangle?

7. $P(3, -2, -3)$, $Q(7, 0, 1)$, $R(1, 2, 1)$

$d(P,Q) = \sqrt{(3-7)^2 + (-2-0)^2 + (-3-1)^2}$

$= \sqrt{16 + 4 + 16} = \sqrt{36} = 6$

$d(P,R) = \sqrt{(3-1)^2 + (-2-2)^2 + (-3-1)^2}$

$= \sqrt{4 + 16 + 16} = \sqrt{36} = 6$

Isosceles

$d(Q,R) = \sqrt{(7-1)^2 + (0-2)^2 + (1-1)^2}$

$= \sqrt{36 + 4} = \sqrt{40} = 2\sqrt{10}$

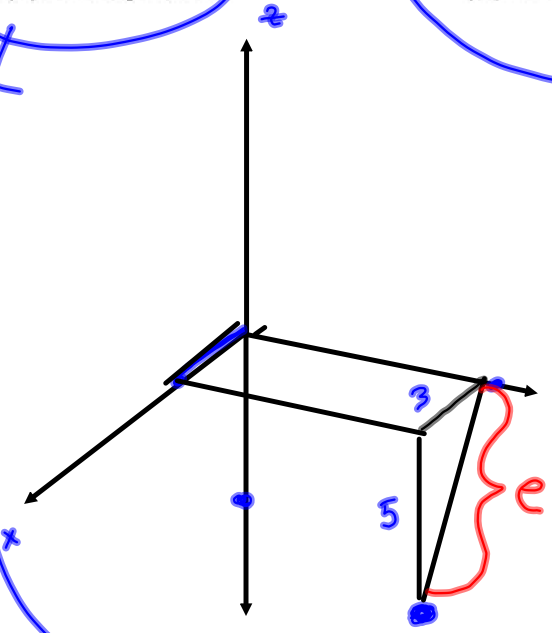
10. Find the distance from $(3, 7, -5)$ to each of the following.

(a) The xy-plane 5 (b) The yz-plane

(c) The xz-plane 7 (d) The x-axis

(e) The y-axis (f) The z-axis

Nearest point on xy-plane is $(3,7,0)$ & so distance is to $(3,7,-5)$



$e = \sqrt{3^2+5^2} = \sqrt{9+25} = \sqrt{34}$

Closest point is $(0,7,0)$ to $(3,7,5)$

$\sqrt{(0-3)^2+(7-7)^2+(0-5)^2} = \sqrt{34}$

12. Find an equation of the sphere with center (2, −6, 4) and radius 5. Describe its intersection with each of the coordinate planes.

$$(x-2)^2 + (y+6)^2 + (z-4)^2 = 25$$

xy-plane: $z=0$

$$(x-2)^2 + (y+6)^2 + 16 = 25$$

$$(x-2)^2 + (y+6)^2 = 9$$

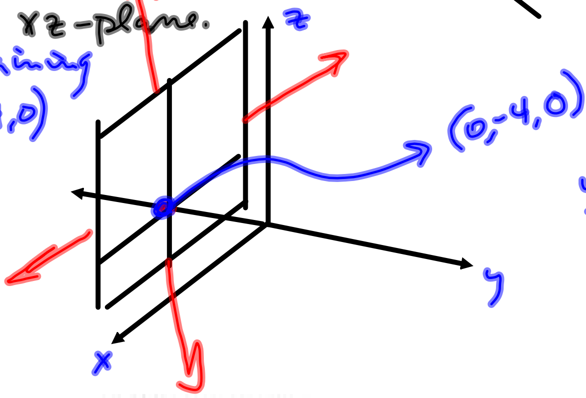
circle of radius 3, centered @ (2, −6)

23–32 Describe in words the region of $\mathbb{R}^3$ represented by the equation or inequality.

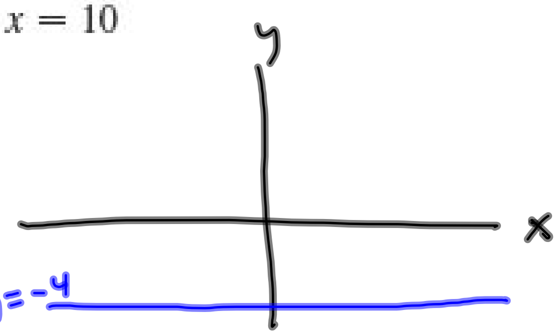
23. $y = -4$

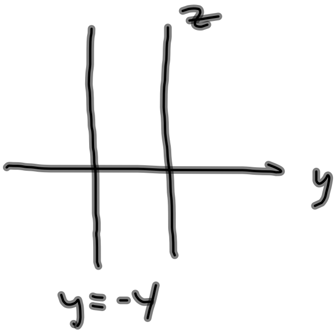
A plane parallel to the xz-plane.

containing $(0,-4,0)$

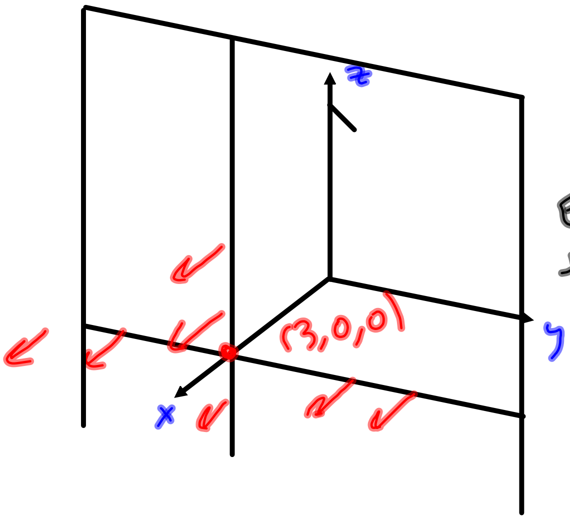


24. $x = 10$





25. $x > 3$

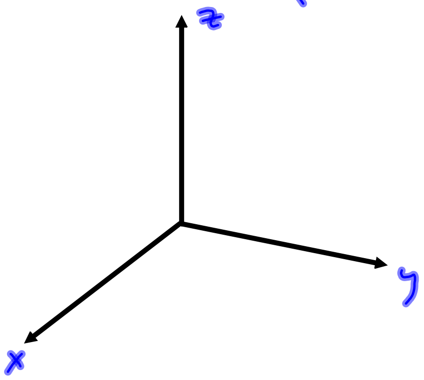


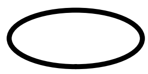
Everything this side of the plane $x=3$

33–36 Write inequalities to describe the region.

34. The solid cylinder that lies on or below the plane $z = 8$ and on or above the disk in the xy-plane with center the origin and radius 2

$\{(x,y,z) \mid x^2+y^2 \le 4,\ 0 \le z \le 8\}$





15–18 Show that the equation represents a sphere, and find its center and radius.

15. $x^2 + y^2 + z^2 - 6x + 4y - 2z = 11$

$x^2 - 6x + 3^2 + y^2 + 4y + 2^2 + z^2 - 2z + 1^2 = 11 + 9 + 4 + 1$

$(x-3)^2 + (y+2)^2 + (z-1)^2 = 25$

$(h, k, \ell) = (3, -2, 1),\ r = 5$

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