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(1) (a) (10pts) $\mathcal{L}_1$: $x = 2+5t,\ y = 3-2t,\ z = 1+t$

$\mathcal{L}_2$: $x = 5-3s,\ y = 1-s,\ z = s$

$x = 2+5t = 5-3s \longrightarrow 3s + 5t = 3$

$y = 3-2t = 1-s \longrightarrow s - 2t = -2$

$z = 1+t = s \longrightarrow s - t = 1$

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 3 & 5 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & -1 & -3 \\ 0 & 8 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -24 \end{array}\right]$$

$0 = -24$?! $\mathcal{L}_1$ & $\mathcal{L}_2$ do not intersect, i.e., they are skew.

(b) (10pts) $\mathcal{L}_1$ & $\mathcal{L}_2$ skew $\longrightarrow \exists$ planes $\mathcal{P}_1, \mathcal{P}_2 \ni \mathcal{L}_1 \subset \mathcal{P}_1$ & $\mathcal{L}_2 \subset \mathcal{P}_2$ and $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$. These are necessarily parallel, with normals that are $\perp$ to $\mathcal{L}_1$ & $\mathcal{L}_2$:

$$\begin{array}{rl} & \bar{u} = \langle 5, -2, 1 \rangle\ 5, -2 \\ \times & \bar{v} = \langle -3, -1, 1 \rangle\ {-3}, -1 \\ \hline & \bar{n} = \langle -1, -8, -11 \rangle \end{array}$$

Now, $P(2,3,1) \in \mathcal{P}_1$ & $Q(5,1,0) \in \mathcal{P}_2$.

Two find the distance from Q to $\mathcal{P}_1$:

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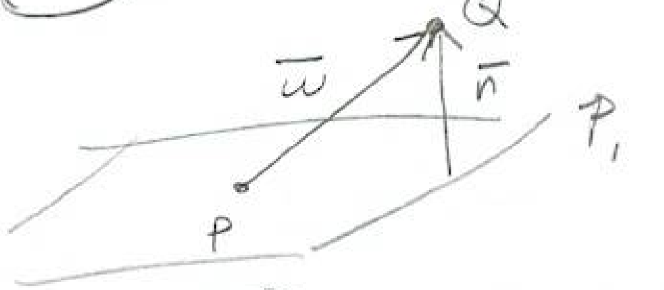
#1b cnt'd



$$\overline{w} = \overrightarrow{PQ} = \langle 5-2, 1-3, 0-1 \rangle = \langle 3,-2,-1 \rangle = \overline{w}$$

$$\text{Distance} = |\text{comp}_{\overline{n}} \overline{w}| = \frac{|\overline{w} \cdot \overline{n}|}{\|\overline{n}\|} = D$$

$$= \frac{|\langle 3,-2,-1 \rangle \cdot \langle -1,-8,-11 \rangle|}{\sqrt{1+64+121}} = \frac{|-3+16+11|}{\sqrt{186}}$$

$$= \frac{24}{\sqrt{186}} = \frac{24\sqrt{186}}{186} = \frac{12\sqrt{186}}{93} = \boxed{\frac{4\sqrt{186}}{31} = D}$$

(2) $\overline{r} = \langle e^t \sin(t), e^t \cos(t), \sqrt{2} e^t \rangle$

(a) (10pts) We find arc length function from $(0,1,\sqrt{2})$

$s(t) = \int_a^t ds$, where a is the value of t corresponding to the point $(0,1,\sqrt{2})$:

$e^t \sin(t) = 0$ ∴ $t=0$ is a sol'n.

$e^t \cos(t) = 1$ ∴ " " " "

$\sqrt{2} e^t = 1$ ∴ " " " "

So $a=2$ works. (It's the only sol'n.)

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2a cntd

$ds = \|\bar{r}'(t)\| dt$, and $\|\bar{r}'(t)\| =$

$= \| \langle e^t \sin t + e^t \cos t, e^t \cos t - e^t \sin t, \sqrt{2} e^t \rangle \|$

$= (e^{2t} \sin^2 t + 2e^{2t} \sin t \cos t + e^{2t} \cos^2 t$
$+ e^{2t} \cos^2 t - 2e^{2t} \sin t \cos t + e^{2t} \sin^2 t + 2e^{2t})^{\frac{1}{2}}$

$= \sqrt{2e^{2t} \sin^2 t + 2e^{2t} \cos^2 t + 2e^{2t}}$

$= \sqrt{2e^{2t} + 2e^{2t}} = \sqrt{4e^{2t}} = 2e^t = \|\bar{r}'(t)\| \Longrightarrow$

$s(t) = \int_0^t 2e^u du = 2e^t - 2e^0 = \boxed{2e^t - 2 = s(t)}$

$\Longrightarrow 2e^t = s+2$

$\Longrightarrow e^t = \frac{s+2}{2}$

$\Longrightarrow \boxed{t = \ln\left(\frac{s+2}{2}\right)} \Longrightarrow$

$\Longrightarrow \bar{r}(t) = \bar{r}(t(s)) = \langle e^{\ln\left(\frac{s+2}{2}\right)} \sin\left(\ln\left(\frac{s+2}{2}\right)\right),$
$e^{\ln\left(\frac{s+2}{2}\right)} \cos\left(\ln\left(\frac{s+2}{2}\right)\right), \sqrt{2} e^{\ln\left(\frac{s+2}{2}\right)} \rangle$

$= \boxed{\langle \frac{s+2}{2} \sin\left(\ln\left(\frac{s+2}{2}\right)\right), \frac{s+2}{2} \cos\left(\ln\left(\frac{s+2}{2}\right)\right), \frac{\sqrt{2}}{2}(s+2) \rangle = \bar{r}(t(s))}$

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2b 10 pts when $s=4$, we have

$$\bar{r}(t(4)) = \left\langle \frac{4+2}{2}\sin\left(\ln\left(\frac{4+2}{2}\right)\right), \frac{4+2}{2}\cos\left(\ln\left(\frac{4+2}{2}\right)\right), \frac{\sqrt{2}}{2}(4+2)\right\rangle$$

$$= \langle 3\sin(\ln(3)), 3\cos(\ln(3)), 3\sqrt{2}\rangle$$

$= \bar{r}(t(4))$ = position corresponding to $s=4$

2c 10 pts $\bar{r} = \langle e^t \sin t, e^t \cos t, \sqrt{2}e^t\rangle$

By previous work,

$$\bar{T} = \frac{\bar{r}'}{\|\bar{r}'\|} = \frac{\langle e^t(\sin t+\cos t), e^t(\cos t - \sin t), \sqrt{2}e^t\rangle}{2e^t}$$

$$= \boxed{\tfrac{1}{2}\langle \cos t+\sin t, \cos t - \sin t, \sqrt{2}\rangle = \bar{T}}$$

$$\bar{N} = \frac{\bar{T}'}{\|\bar{T}'\|} = \frac{\frac{1}{2}\langle -\sin t+\cos t, -\sin t-\cos t, 0\rangle}{\left(\cos^2 t - 2\sin t\cos t + \sin^2 t + \cos^2 t + 2\sin t\cos t + \sin^2 t\right)^{1/2}}$$

$$= \frac{1}{2}\frac{\langle \cos t - \sin t, -(\sin t+\cos t), 0\rangle}{\sqrt{2\cos^2 t + 2\sin^2 t}}$$

$$= \boxed{\frac{1}{2\sqrt{2}}\langle \cos t - \sin t, -(\cos t+\sin t), 0\rangle = \bar{N}}$$

$$\frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

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2 c 10pts $\bar{r} = \langle e^t \sin t, e^t \cos t, \sqrt{2} e^t \rangle$

$\bar{r} = \langle e^t \sin t, e^t \cos t, \sqrt{2} e^t \rangle \Longrightarrow$

$\bar{r}' = \langle e^t(\sin t + \cos t), e^t(\cos t - \sin t), \sqrt{2} e^t \rangle$

$\|\bar{r}'\| = \left(e^{2t}(\sin^2 t + 2\sin t\cos t + \cos^2 t + \cos^2 t - 2\sin t \cos t + \sin^2 t) + 2e^{2t} \right)^{1/2}$

$= \sqrt{e^{2t}(2\cos^2 t + 2\sin^2 t) + 2e^{2t}} = \sqrt{2e^{2t} + 2e^{2t}}$

$= \sqrt{4e^{2t}} = 2e^t \Longrightarrow$

$\bar{T} = \dfrac{\bar{r}'}{\|\bar{r}'\|} = \dfrac{\langle e^t(\sin t + \cos t), e^t(\cos t - \sin t), \sqrt{2} e^t \rangle}{2e^t}$

$\boxed{= \frac{1}{2} \langle \sin t + \cos t, \cos t - \sin t, \sqrt{2} \rangle = \bar{T}}$

$\bar{T}' = \frac{1}{2} \langle \cos t - \sin t, -\sin t - \cos t, 0 \rangle$

$\|\bar{T}'\| = \frac{1}{2}\left(2\cos^2 t + 2\sin^2 t\right)^{\frac{1}{2}} = \frac{\sqrt{2}}{2} \Longrightarrow$

$\bar{N} = \frac{2}{\sqrt{2}} \cdot \frac{1}{2} \langle \cos t - \sin t, -\sin t - \cos t, 0 \rangle$

$\boxed{= \frac{\sqrt{2}}{2} \langle \cos t - \sin t, -\sin t - \cos t, 0 \rangle = \bar{N}}$

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$\vec{B} = \vec{T} \times \vec{N}$

$\frac{1}{2}\langle \sin t + \cos t,\ \cos t - \sin t,\ \sqrt{2} \rangle$ $\sin t + \cos t,\ \cos t - \sin t$

$\times\ \frac{\sqrt{2}}{2}\langle \cos t - \sin t,\ -\sin t - \cos t,\ 0 \rangle$ $\cos t - \sin t,\ -\sin t - \cos t$

$\frac{1}{2} \cdot \frac{\sqrt{2}}{2} \langle \sqrt{2}(\sin t + \cos t),\ \sqrt{2}(\cos t - \sin t),\ -(\sin t + \cos t)^2 - (\sin t - \cos t)^2$

$= \frac{\sqrt{2}}{4}\langle \sqrt{2}(\sin t + \cos t),\ \sqrt{2}(\cos t - \sin t),\ -2 \rangle$

$= \langle \frac{1}{2}(\sin t + \cos t),\ \frac{1}{2}(\cos t - \sin t),\ \frac{-\sqrt{2}}{2} \rangle = \vec{B}$

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3 10 pts $z = x^2 - xy + 3y^2 \Longrightarrow$

$dz = (2x - y)\,dx + (-x + 6y)\,dy$

$(3,-1) \longrightarrow (2.96, -.95) \Longrightarrow$

$\Delta x = -.04,\ \Delta y = .05 \Longrightarrow$

$dz = (2(3)+1)(-.04) + (-3 + 6(-1))(.05)$

$= (7)(-.04) + (-9)(.05)$

$= -.28 - .45 = \boxed{-.73 = dz}$

$-\Delta z = z(3,-1) - z(2.96, -.95)$

$= 3^2 - (3)(-1) + 3(-1)^2 - \left(2.96^2 - (2.96)(-.95) + 3(.95)^2\right)$

$= 9 + 3 + 3 - (14.2811) = 15 - 14.2811 = .7189$

$\Longrightarrow \boxed{\Delta z = -.7189}$

I did $z(x,y) - z(x+\Delta x, y+\Delta y)$, which is backwards.

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4) $\bar{F} = \langle -2yz, y, 3x \rangle$, $S = \{(x,y,z) \mid z = 4 - x^2 - y^2,\ 0 \le z \le 4\}$, oriented upward.

We verify STOKES' Theorem:

a) 10 pts $\int_C \bar{F} \cdot d\bar{r} = \int_C \bar{F}(\bar{r}(t)) \cdot \bar{r}'(t)\, dt$

where C = circle of radius 2 in the xy-plane, embedded in 3-space, i.e.

$\bar{r}(\theta) = \langle 2\cos\theta, 2\sin\theta, 0 \rangle \quad 0 \le \theta \le 2\pi$

$\Rightarrow \bar{r}'(\theta) = \langle -2\sin\theta, 2\cos\theta, 0 \rangle$

$\bar{F} \cdot \bar{r}' = \langle -2yz, y, 3x \rangle \cdot \langle x', y', z' \rangle$

$= -2yzx' + yy' + 3xz'$

$= -2(2\sin\theta)(0)x' + 4\sin\theta\cos\theta + 3(2\cos\theta)(0)$

$= 4\sin\theta\cos\theta$

$\Rightarrow \int_C \bar{F} \cdot d\bar{r} = \int_0^{2\pi} 4\sin\theta\cos\theta\, d\theta \; \boxed{= 0}$!

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(4b) $\iint_S \text{curl}\,\bar{F}\cdot d\bar{S} = \iint \text{curl}\,\bar{F}\cdot(\bar{r}_x \times \bar{r}_y)\,dA$

$\langle x, y, z\rangle\; x, y$

$\times \langle -2yz, y, 3x\rangle\; -2yz, y$

$\langle 0, -2y-3, 2z\rangle = \text{curl}\,\bar{F}$

$\bar{r}_x = \langle 1, 0, -2x\rangle\; 1, 0$

$\times\; \bar{r}_y = \langle 0, 1, -2y\rangle\; 0, 1$

$\langle 2x, 2y, 1\rangle$

$\iint_S \text{curl}\,\bar{F}\cdot d\bar{S} = \iint_S \langle 0, -2y-3, 2z\rangle\cdot\langle 2x, 2y, 1\rangle\,dA$

$= \iint_S (-4y^2 - 6y + 2z)\,dA$

Let $x = r\cos\theta$, $y = r\sin\theta$, $z = 4 - x^2 - y^2 = 4 - r^2$

Then we have, by change-of-variables,

$\int_0^{2\pi}\int_0^2 \left(-4r^2\sin^2\theta - 6r\sin\theta + 2(4-r^2)\right) r\,dr\,d\theta$

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4b cont'd

$$= \int_0^{2\pi}\int_0^2 -4r^3\sin^2\theta - 6r^2\sin\theta + 8 - 2r^3\,)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[-r^4\sin^2\theta - 2r^3\sin\theta + 8r - \tfrac{1}{2}r^4\right]_0^2 d\theta$$

$$= \int_0^{2\pi}\left[-16\sin^2\theta - 16\sin\theta + 16 - 8\right] d\theta$$

$$= \int_0^{2\pi}\left[-8 + 8\cos(2\theta) - 16\sin\theta + 8\right] d\theta$$

$$= \int_0^{2\pi}\left(8\cos(2\theta) - 16\sin\theta\right) d\theta = \boxed{0}$$

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(5) ~~10pts~~ $\bar{v} = \langle x, y^2, 2yz \rangle$

$S = \{(x,y,z) \mid x^2+y^2=4,\ 0 \le z \le 1\}$

This question is poorly posed. Clearly, the intention was to include top & bottom, or Divergence Theorem doesn't apply.

Top: $\{(x,y,z) \mid x^2+y^2 \le 4,\ z=1\} = S_1$

Bottom: $\{(x,y,z) \mid x^2+y^2 \le 4,\ z=0\} = S_2$

(a) $\rho \iint_S \bar{v} \cdot d\bar{S} = 5 \iint_S \bar{v} \cdot d\bar{S}$

~~(a)~~ 10pts $\bar{r} = \langle r\cos\theta, r\sin\theta, 1 \rangle \quad 0 \le r \le 2,\ 0 \le \theta \le 2\pi$

$\bar{r}_r = \langle \cos\theta, \sin\theta, 0 \rangle \quad \cos\theta, \sin\theta$

$\times \quad \bar{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle \quad -r\sin\theta, r\cos\theta$

$\langle 0, 0, r\cos^2\theta + r\sin^2\theta \rangle = \langle 0, 0, r \rangle$

$\bar{v}(\bar{r}(r,\theta)) = \langle r\cos\theta, r^2\sin^2\theta, 2r\sin\theta \cdot 0 \rangle$

$= \langle r\cos\theta, r^2\sin^2\theta, 0 \rangle$

$\int_0^{2\pi}\int_0^2 \langle r\cos\theta, r^2\sin^2\theta, 0 \rangle \cdot \langle 0,0,r \rangle\, dr\, d\theta = \boxed{0}$

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S_2 is BOTTOM. Its normal is $\bar{n} = \langle 0,0,-r\rangle$

or rather, its $\bar{r}_r \times \bar{r}_\theta$ is $\langle 0,0,-r\rangle$

$\rho \iint_{S_2} \bar{v} \cdot d\bar{S} = 0$, also. It remains to

find $\rho \iint_{S_3} \bar{v} \cdot d\bar{S}_3$ where S_3 is the sidewall.

$\bar{r} = \langle 2\cos\theta, 2\sin\theta, z\rangle$ using cylindrical

coordinates

$\bar{r}_\theta = \langle -2\sin\theta, 2\cos\theta, 0\rangle \quad -2\sin\theta, 2\cos\theta$

$\bar{r}_z = \langle 0, 0, 1\rangle \quad 0, 0$

$\langle 2\cos\theta, 2\sin\theta, 0\rangle \longrightarrow$

$$\rho \int_0^{2\pi}\int_0^1 \bar{v} \cdot (\bar{r}_\theta \times \bar{r}_z)\, d\theta\, dz =$$

$$5 \int_0^{2\pi}\int_0^1 \langle 2\cos\theta, 4\sin^2\theta, 4\sin\theta\, z\rangle \cdot \langle 2\cos\theta, 2\sin\theta, 0\rangle$$
$dz\, d\theta$

$$= \int_0^{2\pi} 5 \int_0^1 (4\cos^2\theta + 8\sin^3\theta)\, dz\, d\theta = \int_0^{2\pi} 5\left[4\cos^2\theta + 8\sin^3\theta\right] d\theta$$

$$= 5\int_0^{2\pi} \left[2 + 2\cos(2\theta) + 8\sin\theta(1-\cos^2\theta)\right] d\theta$$

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$$= 5\int_0^{2\pi}\left[2 + 2\cos(2\theta) + 8\sin\theta - 8\cos^2\theta\sin\theta\right]d\theta$$

$$\boxed{= 20\pi}$$

5b 10pts

$$\operatorname{div}\vec{v} = 1 + 2y + 2y = 4y + 1$$

$$5\iiint_E (4y+1)\,dV =$$

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 5\int_0^1 (4y+1)\,dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 5\left[(4y+1)z\right]_0^1 dy\,dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4y+1)\,dy\,dx$$

$$= \int_0^{2\pi} 5\int_0^2 (4(2\cos\theta)+1)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} 5\left[(8\cos\theta + 1)\right]\left[\frac{r^2}{2}\right]_0^2 = 5\int_0^{2\pi}\left[16\cos\theta + 2\right]d\theta$$

$$\boxed{= 20\pi\ \frac{kg}{s}}$$