MAT 203
100 Points

Test 3 – Fall, 20
Covers Chapter 15

Name KEY
NO GRAPHING CALCULATORS!!!

Do all your work and submit answers with your work, on the separate paper provided. Organize your work for efficient grading and feedback. Leave a margin, especially in the top left, where the staple goes!

Make sure you have everything set up before doing any messy calculations. Draw more pictures than you want to.

1. (20 pts) Use Fubini to calculate the double integral $\iint_R \cos(\theta)\sin(\phi)\, dA$, where $R$ is the rectangle

$$\left\{(\theta,\phi)\,|\,0\le\theta\le\frac{\pi}{3}, 0\le\phi\le\frac{\pi}{4}\right\}.$$

2. Polar and Cylindrical Coordinates.

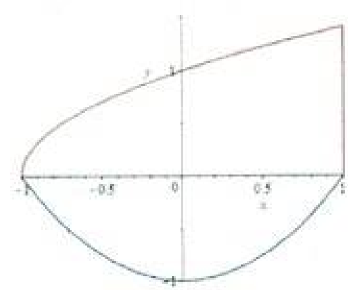
   a. (20 pts) Use polar coordinates and a double integral to find the total mass of the lamina that is the portion of a circle of radius $r = 3$ centered at the origin that lies in the first quadrant, if the mass density function $\rho(x,y)$ at $Q(x,y)$ is proportional to the square of $Q$'s distance from the origin.

   b. (5 pts **bonus**) Write a double integral for the previous problem in *rectangular* coordinates.

   c. (10 pts) Use cylindrical coordinates and a triple integral to find the volume of the solid bounded by the paraboloid $z = 9 - x^2 - y^2$ in the first octant.

   d. (5 pts **bonus**) Write a triple integral for the previous problem in *rectangular* coordinates.

3. SET UP (Do Not Evaluate) the double integral $\iint_R x\, dA$ over the region $R$ bounded by $y = x^2 - 1, y = \sqrt{x+1}, x = 1$. Do this by viewing $R$ as a...




   a. (20 pts) ... Type I region.

   b. (10 pts) ... Type II region.

4. (20 pts) Evaluate $\iiint_R \sqrt{x^2+y^2+z^2}\, dV$, where $R$ lies above the cone $z = \sqrt{x^2+y^2}$ and between the spheres $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 4$. Use spherical coordinates.

**Bonus Section for sit-down test:**

1. (10 pts) Find the Jacobian of a change of variables that maps the parallelogram bounded by the inequalities $6 \le 3x+2y \le 12, 15 \le 5x-3y \le 30$ into a nice rectangle in the $uv$-plane.

2. (10 pts) Write as many of the 5 other iterated integrals that are equivalent to the iterated integral $\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)\,dz\,dy\,dx$ as you can find. 2 points apiece.

203 TEST 3 FALL '20

(1) $$\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{3}} \cos\theta \sin\phi \, d\theta \, d\phi$$

$$= \int_0^{\frac{\pi}{4}} \sin\phi \, d\phi \int_0^{\frac{\pi}{3}} \cos\theta \, d\theta$$

20pts

$$= \left[-\cos\phi\right]_0^{\frac{\pi}{4}} \left[\sin\theta\right]_0^{\frac{\pi}{3}} = \left(-\cos\frac{\pi}{4} + \cos 0\right)\left(\sin\frac{\pi}{3} - \sin 0\right)$$

$$= \left(-\frac{1}{\sqrt{2}} + 1\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}\left(\frac{-\sqrt{2}}{2} + \frac{2}{2}\right) = \boxed{\frac{2\sqrt{3} - \sqrt{6}}{4}}$$

(2a) $$K\int_0^{\frac{\pi}{2}}\int_0^{3} r^2 \, r \, dr \, d\theta = \int_0^{\frac{\pi}{2}} K\left[\frac{r^4}{4}\right]_0^3 d\theta$$

20pts

$$= \int_0^{\frac{\pi}{2}} K\left[\frac{81}{4}\right] d\theta = \frac{81}{4}K\left[\frac{\pi}{2}\right] = \boxed{\frac{81\pi}{8}K}$$

(2b) 5pts $$\int_0^3 \int_0^{\sqrt{9-x^2}} K(x^2+y^2) \, dy \, dx$$

$$= \int_0^3 \left[x^2 y + \frac{y^3}{3}\right]_0^{\sqrt{9-x^2}} dx = \int_0^3 \left[x^2\sqrt{9-x^2} + \frac{1}{3}(9-x^2)^{\frac{3}{2}}\right] dx$$


$\frac{x}{3} = \sin\theta$

$x = 3\sin\theta$

$\sqrt{9-x^2} = 3\cos\theta$

$dx = 3\cos\theta \, d\theta$

$$= \int_{x=0}^{x=3} 9\sin^2\theta\,(3\cos\theta) + \frac{1}{3}\cos^3\theta \,\big)\, 3\cos\theta \, d\theta$$

$$= \int_{0=x}^{3=x} 81\cos^2\theta \sin^2\theta \, d\theta$$

$$+ \int_{x=0}^{x=3} \cos^4\theta \, d\theta$$

203 T3

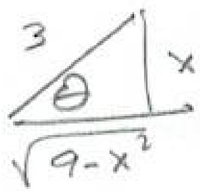
1.5

(2b) $\int_0^3 \int_0^{\sqrt{9-x^2}} (x^2+y^2)\, dy\, dx$

$= \int_0^3 \left[ x^2 y + \frac{1}{3} y^3 \right]_0^{\sqrt{9-x^2}} dx$

Never did find my mistake in the evaluation.

$= \int_0^3 \left[ x^2 \sqrt{(9-x^2)} + \frac{1}{3} (9-x^2)^{\frac{3}{2}} \right] dx$




$x = 3\sin\theta \longrightarrow dx = 3\cos\theta\, d\theta$

$\sqrt{9-x^2} = 3\cos\theta$

$= \int_{0=x}^{3=x} (3^2 \sin^2\theta)(3\cos\theta) + \frac{1}{3}(3\cos\theta)^3 \Big) (3\cos\theta\, d\theta)$

$= \int_{x=0}^{x=3} \Big( 27\sin^2\theta\cos\theta + \frac{3^3}{3}\cos^3\theta \Big)(3\cos\theta\, d\theta)$

$= \int_{x=0}^{x=3} \big( 81\sin^2\theta\cos^2\theta + 27\cos^4\theta \big)\, d\theta$

$\sin^2\theta(1-\sin^2\theta)\cos\theta = \sin^2\theta\cos\theta - \sin^4\theta\cos\theta$

$\left(\frac{\cos 2\theta + 1}{2}\right)^2 = \frac{\cos^2 2\theta + 2\cos 2\theta + 1}{4}$

$= \frac{1}{4}\left[ \frac{\cos 4\theta + 1}{2} + 2\cos 2\theta + 1 \right] = \frac{1}{4}\left[ \frac{\cos 4\theta}{2} + 2\cos 2\theta + \frac{3}{2} \right]$

OK

203 T3 1.75

$\sin^2\theta \cos^2\theta$

$= \left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cos 2\theta}{2}\right)$

$= \frac{1}{4}\left[1 - \cos^2 2\theta\right]$

$= \frac{1}{4}\left[1 - \frac{1+\cos 4\theta}{2}\right] = \frac{1}{4} - \frac{1}{8} - \frac{1}{8}\cos 4\theta$

$= \frac{1}{8} - \frac{1}{8}\cos 4\theta$

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$= \frac{81}{8}\int_0^{\frac{\pi}{2}}(1-\cos 4\theta)\,d\theta + \frac{27}{8}\int_0^{\frac{\pi}{2}}(\cos 4\theta + 4\cos 2\theta + 3)\,d\theta$

$= \frac{81}{8}\left[0 - \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{2}} + \frac{27}{8}\left[\frac{1}{4}\sin 4\theta + 2\sin 2\theta + 3\theta\right]_0^{\frac{\pi}{2}}$

$= \frac{81}{8}[0] + \frac{27}{8}\left[\frac{3\pi}{2}\right] = \frac{81\pi}{16}$ ?!

I can't get this any closer, no matter what I do!

203 $\sqrt{3}$ 

(2b) Cont'd

$$= 81 \int_{x=0}^{x=3} \left( \frac{1}{8} - \frac{1}{8}\cos(4\theta) \right) d\theta = \frac{81}{8} \left[ \theta - \sin(4\theta) \right]_{x=0}^{x=3}$$

$$-\frac{(\cos 2\theta + 1)(\cos 2\theta - 1)}{4} = -\frac{\cos^2(2\theta) - 1}{4}$$

$$= +\frac{1}{4} - \frac{\cos^2(2\theta)}{4} = \frac{1}{4} - \left( \frac{\cos(4\theta) + 1}{8} \right)$$

$$= \frac{1}{4} - \frac{1}{8} - \frac{1}{8}\cos(4\theta)$$

$$= \frac{81}{8} \left[ \arcsin\left(\frac{x}{3}\right) - \frac{x}{3} \right]_0^3$$

$$= \frac{81}{8} \Big[$$

$$= \frac{81}{8} \left[ \frac{\pi}{2} - 1 - (0 - 0) \right]$$

$x = 3\sin\theta$

$\frac{x}{3} = \sin\theta$

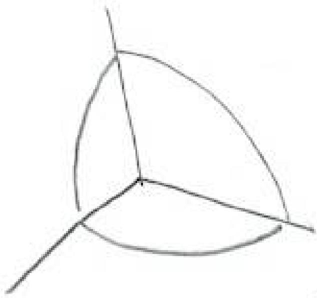
$\arcsin\left(\frac{x}{3}\right) = \theta$

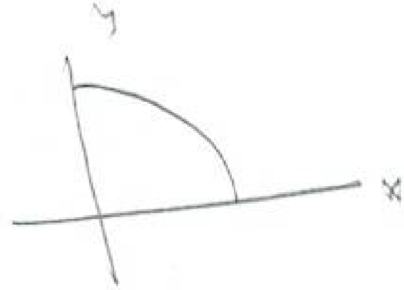
$$= \frac{81}{8} \left[ \frac{\pi}{2} - 1 \right]$$

203 T3



2c 10 pts






$$\int_0^{\frac{\pi}{2}}\int_0^3\int_0^{9-r^2} r\,dz\,dr\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}\int_0^3 rz\Big]_0^{9-r^2} dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^3 r(9-r^2)\,dr\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}\int_0^3 [9r-r^3]\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \Big[\frac{9}{2}r^2-\frac{1}{4}r^4\Big]_0^3 d\theta$$

$$=\int_0^{\frac{\pi}{2}}\Big(\frac{81}{2}-\frac{81}{4}\Big)\,d\theta = \frac{81}{4}\Big[\frac{\pi}{2}\Big] = \frac{81\pi}{8}$$

2d 5 pts $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{9-x^2y^2} dz\,dy\,dx$

203 T3 (4)

3 a 20 pts

$$\int_{-1}^{1}\int_{x^2-1}^{\sqrt{x+1}} x \, dy \, dx$$

b 10 pts

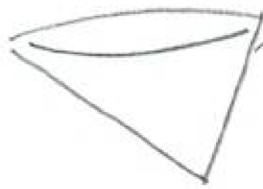
$$\int_{-1}^{0}\int_{-\sqrt{y+1}}^{\sqrt{y+1}} x \, dx \, dy + \int_{0}^{\sqrt{2}}\int_{y^2-1}^{1} x \, dx \, dy$$

4 20 pts

$\iiint_R \sqrt{x^2+y^2+z^2}$ R is above $z=\sqrt{x^2+y^2}$

and between $x^2+y^2+z^2=1$ & $x^2+y^2+z^2=4$

$1 \le \rho \le 2$



$z=\sqrt{x^2+y^2}$

in $y=0$ plane:

$z=\pm x$

That's a $45^\circ$ / $\frac{\pi}{4}$ angle

So $0 \le \phi \le \frac{\pi}{4}$

Go full circle: $0 \le \theta \le 2\pi$

$$\int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_1^2 \rho \cdot \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \left(\int_0^{\frac{\pi}{4}} \sin\phi \, d\phi\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_1^2 \rho^3 d\rho\right)$$

$$= \left(-\frac{\sqrt{2}}{2}+\frac{2}{2}\right)\left(2\pi\right)\left(\frac{2^4-1}{4}\right) = \boxed{\frac{15\pi}{4}\left(2-\sqrt{2}\right)}$$

203 BONUS T3 5

B1 Let $u = 3x + 2y$ TIMES $-5$ $\quad -15x - 10y = -5u$

$v = 5x - 3y$ TIMES 3 $\quad 15x - 9y = 3v$

$-19y = -5u + 3v$

$y = \frac{5u - 3v}{19}$

$\begin{array}{r} 38 \\ 19 \\ \hline 57 \end{array}$

$$3x + 2y = 3x + 2\left(\frac{5u-3v}{19}\right) = 3x + \frac{10u - 6v}{19}$$

$$= \frac{57x + 10u - 6v}{19} = \frac{19u}{19}$$

$\Rightarrow 57x = 9u + 6v$

$$x = \frac{9u+6v}{57} = \boxed{\frac{3u+2v}{19} = x}$$

$\bar{r} = \frac{1}{19} < 3u + 2v, 5u - 3v >$

$\bar{r}_u = \frac{1}{19} < 3, 5 > \longrightarrow \frac{1}{19^2}(-9 - 10) = \frac{1}{19^2}(-19)$

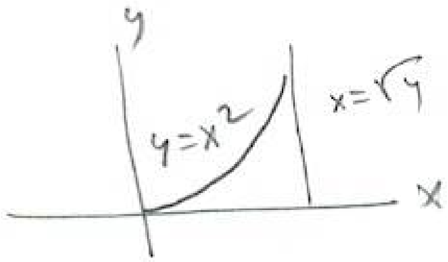
$\bar{r}_v = \frac{1}{19} < 2, -3 >$

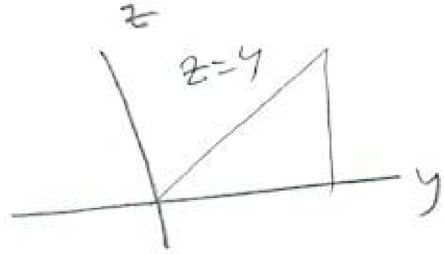
$= -\frac{1}{19} \Rightarrow \boxed{\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{19}}$

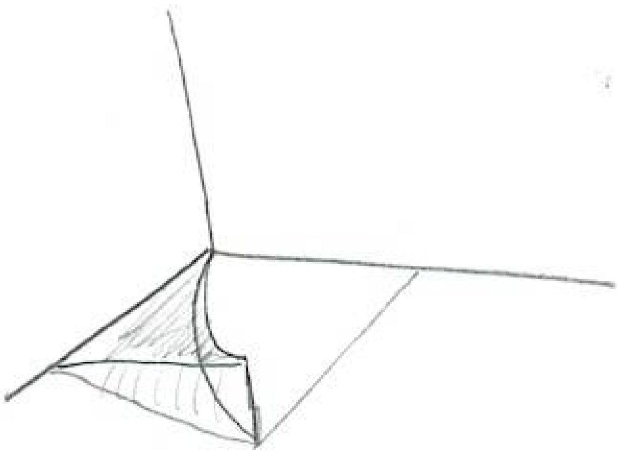
203 T3 BONUS

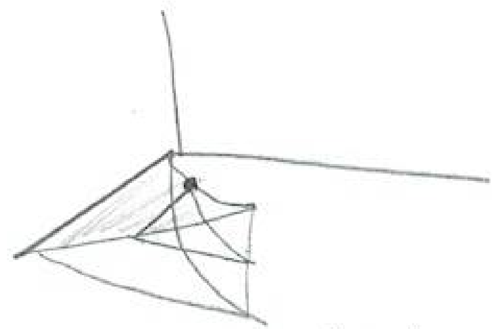
6

B2 $\displaystyle\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)\,dz\,dy\,dx$





$$\int_0^1\int_{\sqrt{y}}^1\int_0^y dz\,dx\,dy$$

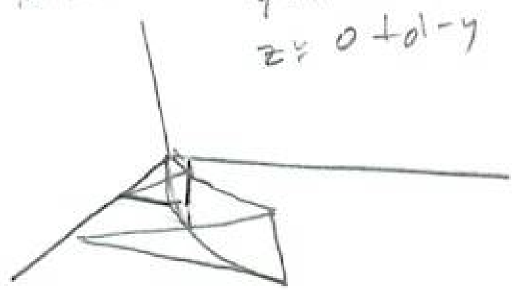
x: $\sqrt{y}$ to 1

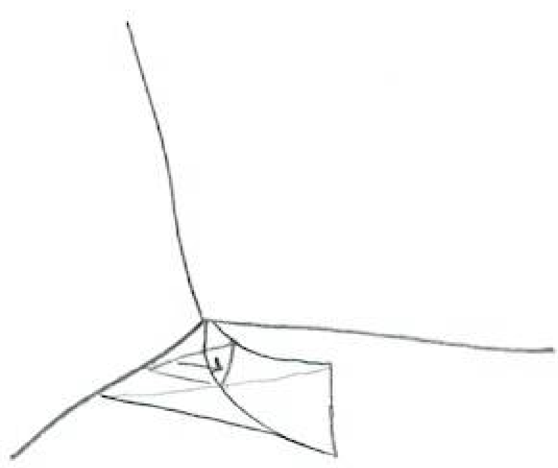
$$\int_0^1\int_{\sqrt{z}}^1\int_z^{x^2} dy\,dx\,dz$$

$$\int_0^1\int_0^{x^2}\int_z^{x^2} dy\,dz\,dx$$

y: 1-z to $x^2$

z: 0 to 1-y



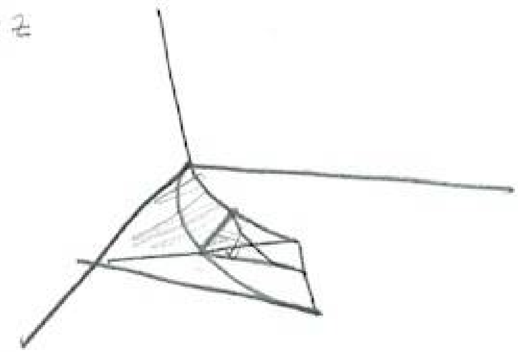


203 BONUS 

B2 cnt'd

$$\int_0^1 \int_z^1 \int_{\sqrt{y}}^1 dx\,dy\,dz$$

$y = x^2 = z$



y from $1-z$ to $\sqrt{x}$

x starts at $\sqrt{y} = \sqrt{1-z}$

to 1

$$\int_0^1 \int_0^{y} \int_{\sqrt{y}}^1 dx\,dz\,dy$$