MAT 203 Test 1 – Fall, 2020 Name KEY

100 Points Covers Chapters 12, 13 NO GRAPHING CALCULATORS!!!

Do all your work and submit answers with your work, on the separate paper provided. Organize your work for efficient grading and feedback. Leave a margin, especially in the top left, where the staple goes!

Leave space between problems. No prizes for saving paper, here. Figure this stuff out, and use your smarts to plant trees! Only use one column of work. Don't start a 2nd column to save paper. ALL I WANT ON THIS PAGE IS YOUR NAME.

1. (10 pts) Find parametric equations and a vector equation for the line of intersection between the two planes:
   $P_1: x+2y-z=7$
   $P_2: 2x+3y+z=11$

2. Essential concepts for math in 3-D: We'll be working with the 4 points $A(1,2,1)$, $B(-3,-2,1)$, $C(2,3,2)$ and $D(7,-1,6)$.

   a. (5 pts) (Line segment) Write the equation of the line segment between $A(1,2,1)$ and $B(-3,-2,1)$.

   b. (5 pts) (Line) Form the vector $\overline{u}=\overrightarrow{AB}$ and find a vector equation for the line containing the points $A(3,-1,2)$ and $B(-3,-2,1)$.
   $\rightarrow A(1,2,1)$, $B(-3,-2,1)$

   c. (5 pts) (Vector Equation of Plane) Form the vector $\overline{v}=\overrightarrow{AC}$, using A and C from part a. Then write a vector equation for the plane containing A, B and C. Sketch the plane using its intercepts.

   d. (5 pts) (Area of Parallelogram) Find the area of the parallelogram defined by the vectors $\overline{u}$ and $\overline{v}$.

   e. (10 pts) (Volume of a Parallelepiped) Form the vector $\overline{w}=\overrightarrow{AD}$, where $\boxed{D(-2,-2,1)}$ is another point. And find the volume of the parallelepiped defined by the 3 edges, $\overline{u}, \overline{v}$, and $\overline{w}$. $\rightarrow D(7,-1,6)$

3. Distance Problems:

   a. (10 pts) Use the identity $\|\overline{u}\times\overline{v}\|=\|\overline{u}\|\|\overline{v}\|\sin(\theta)$ and a big, clear sketch to show that the distance from a point to a line is $d=\dfrac{\|\overline{u}\times\overline{v}\|}{\|\overline{v}\|}$

   b. (10 pts) Let $\overline{r}(0)=\overline{r}_0=\langle 10,-1,7\rangle$ and $\overline{v}=\langle 1,1,1\rangle$. Find the distance from the point $E(7,8,9)$ to the line L: $\overline{r}(t)=\overline{r}_0+t\overline{v}$.

   c. (10 pts) Show that the distance D from the point $Q(x_0,y_0,z_0)$ to the plane $P: ax+by+c=d$ is given by $\dfrac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$. I expect a big, clear sketch that supports the objects (vectors) you define in order to make your argument.

   d. (10 pts) Find the distance from the point $Q(-1,10,20)$ to the plane $P: 3x-2y+5z=11$.

4. (10 pts) Consider the quadric surface $576x^2 - 324y^2 + 144z^2 = 5184$. Show its traces in the planes $x = k, y = k, z = k$ for different choices (at least 2 each) of k and project those into the $yz-$, $xz-$, and $xy-$ planes, respectively. In other words, sketch the projections into the $yz-$, $xz-$, and $xy-$ planes. Then sketch the surface in 3-D.

5. Let $\overline{r}(t) = \langle t, \sin(t), \cos(t) \rangle$.
   a. (5 pts) Find the Unit Tangent $\overline{T}$ and Unit Normal $\overline{N}$, as functions of t.
   b. (5 pts) Evaluate your answers to part a at $t = \frac{\pi}{6}$ *and* find the Unit Binormal $\overline{B}$ at $t = \frac{\pi}{6}$. In other words, find $\overline{T}\left(\frac{\pi}{6}\right), \overline{N}\left(\frac{\pi}{6}\right)$, and $\overline{B}\left(\frac{\pi}{6}\right)$.
   c. (5 pts) Sketch the graph of $\overline{r}(t)$, for $t \in [0, \pi]$. Then draw the TNB frame that you found symbolically in part b. That is, show the unit tangent, normal and binormal in your sketch.

6. (5 pts) Give a verbal description of the statement $\kappa = \left|\frac{d\overline{T}}{ds}\right|$. What is it? What does it mean? What's our shortcut for calculating it, in terms of $\overline{r}(t)$?

BONUS: Answer up to 3 for up to 15 points.

7. (5 pts) Find the curvature of the vector function $\overline{r}(t) = \langle 2\sin(t), 2\cos(t), 1 \rangle$. Relate this to the radius of the osculating circle. This is one you can do by inspection, if you have some intuition.

8. (5 pts) Simplify the derivative: : $\frac{d}{dx} \int_0^{\sin(x)} \frac{\tau^5 + 4\tau^3}{\left(\sqrt{6\tau^3} + \cos^2(2\tau)\right)} d\tau$

9. (5 pts) A projectile is fired with an initial velocity of 500 m/s and an angle of elevation of 45^0. Find its maximum height, range, and speed at which it strikes the ground. And no, our model doesn't work on a spinning planet with spinning projectile and air resistance. It's actually quite complicated, if you want to be *that* precise. But we'd do great with this stuff on the moon.

10. Find the position $\overline{r}(t)$ and velocity $\overline{v}(t)$, given an acceleration function $\overline{a}(t) = \left\langle \frac{1}{t^2+1}, \sec(t), t \right\rangle$, and given $\overline{r}(0) = \langle 2, 3, 1 \rangle$ and $\overline{v}(0) = \langle 3, 2, 1 \rangle$.

203 TEST 1 FALL '20

1

10 pts

$x + 2y - z = 7$

$2x + 3y + z = 11$

See Dylan's approach.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 2 & 3 & 1 & 11 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 0 & -1 & 3 & -3 \end{array}\right]$$

$1 \quad -3 \quad 3$

$y - 3z = 3$

$y = 3z + 3$

$x + 2y + z$

$= x + 2(3z + 3) - z = 7$

$x + 6z + 6 - z = 7$

$x + 5z = 1$

$x = -5z + 1$

$$\boxed{x = -5t + 1,\ y = 3t + 3,\ z = t}$$

$$\boxed{\bar{r}(t) = \bar{r}_0 + \bar{v}t,} \text{ where}$$

$$\boxed{\bar{r}_0 = \langle 1, 3, 0 \rangle,\ \bar{v} = \langle -5, 3, 1 \rangle}$$

203 TEST 1

2

(2) $A(1,2,1)$, $B(-3,-2,1)$, $C(2,3,2)$, $D(7,-1,6)$

(a) $\boxed{\bar{a} = \langle 1,2,1\rangle,\ \bar{b} = \langle -3,-2,1\rangle}$

Segment from A to B is

(5pts) $\boxed{(1-t)\,\bar{a} + t\bar{b}}$ $0 \le t \le 1$ STOP!

$\bar{a} - t\bar{a} + t\bar{b} = \langle 1,2,1\rangle - t\langle 1,2,1\rangle + t\langle -3,-2,1\rangle$

OR $= \boxed{\langle 1,2,1\rangle + t\langle -4,-4,0\rangle}$

OR $\langle -4t+1, -4t+2, 1\rangle$

Alternates: $(1-t)\bar{a} + t\bar{b}$, w/o restriction on t.

(b) (5pts) $\overrightarrow{AB} = \bar{u} = \langle -4,-4,0\rangle$

l: $\langle 1,2,1\rangle + t\langle -4,-4,0\rangle$

OR $\langle -3,-2,1\rangle + t\langle -4,-4,0\rangle$

Vertical plane.

(c) (5pts) $\overrightarrow{AC} = \bar{v} = \langle 1,1,1\rangle$

$$\begin{array}{rl} \bar{u} & \langle -4,-4,0\rangle, -4, -4 \\ \times\ \bar{v} & \langle 1,1,1\rangle, 1, 1 \\ \hline \bar{n} = & \langle -4,4,0\rangle \equiv \bar{n} \end{array}$$

$\bar{n} \cdot \langle x-1, y-2, z-1\rangle$

$= \langle -4,4,-4\rangle \cdot \langle x-1, y-2, z-1\rangle$

$= -4(x-1) + 4(y-2)$

$= -4x + 4 + 4y - 8 = 0$

$\langle 1,2,1\rangle + t\bar{u} + s\bar{v}$

$(s,t) \in \mathbb{R}\times\mathbb{R}$

$\longrightarrow \boxed{-4x + 4y = 4}$

$-x + y = 1$

$(-1,0,0)$ $(0,1,0)$

203 T1

2 d 5pts Area of parallelogram

is $\|\bar{u}\times\bar{v}\| = \|\langle -4, 4, 0\rangle\|$

$= \sqrt{2(16)} = \boxed{4\sqrt{2} = \text{Area}}$

e 10pts Volume of parallelepiped is

$|\bar{w}\cdot(\bar{u}\times\bar{v})| = |\langle 6,-3,5\rangle \cdot \langle -4,4,0\rangle|$

$= |-24-12+0| = \boxed{36 = \text{Volume}}$

$\bar{w} = \langle 7-1, -1-2, 6-1\rangle$

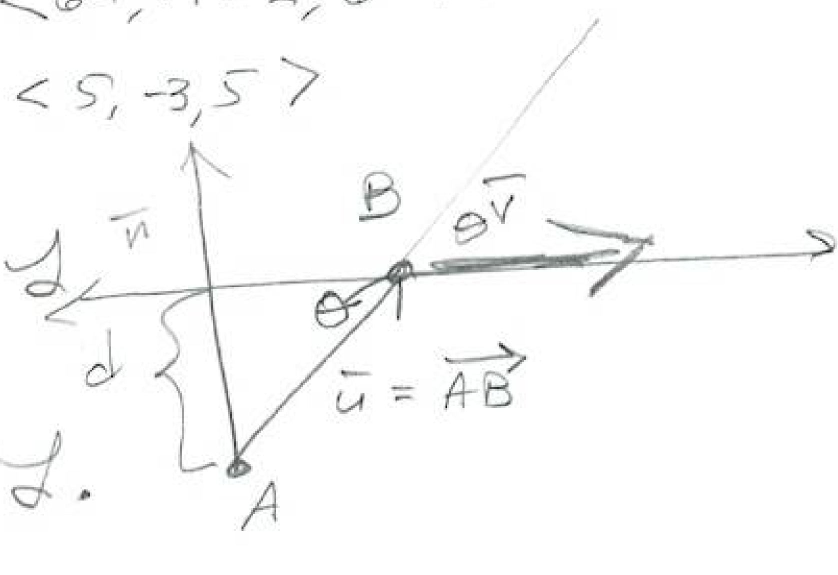
$= \langle 6, -3, 5$

where $\bar{w} = \overrightarrow{AD} = \langle 6-1, -1-2, 6-1\rangle$

$= \langle 5, -3, 5\rangle$

3 a 10pts

$\bar{v}$ is direction vector for line $\mathcal{L}$.



203 T1

(4)

\#3a cnt'd

From pic, we want $d = |\text{comp}_{\bar{n}} \bar{u}|$

$= \frac{|\bar{n} \cdot \bar{u}|}{\|\bar{n}\|}$ we don't <u>Know</u> $\bar{n}$!

So, $\frac{d}{\|\bar{u}\|} = \sin\theta$. From

$\|\bar{u} \times \bar{v}\| = \|\bar{u}\|\|\bar{v}\| \sin\theta$, we obtain

$\sin\theta = \frac{\|\bar{u} \times \bar{v}\|}{\|\bar{u}\|\|\bar{v}\|} \longrightarrow$

$\frac{d}{\|\bar{u}\|} = \frac{\|\bar{u} \times \bar{v}\|}{\|\bar{u}\|\|\bar{v}\|} \longrightarrow$

$\boxed{d = \frac{\|\bar{u} \times \bar{v}\|}{\|\bar{v}\|}}$

$\bar{v} = \langle 1,1,1 \rangle$

$M(10,-1,7)$

3b 10pts $\mathcal{L}: \bar{r} = \langle 10,-1,7 \rangle + t\langle 1,1,1 \rangle$

Distance to $E(7,8,9)$:

2/210
 109

Form $\bar{u} = \overrightarrow{EM} = \langle 3,-9,-2 \rangle \longrightarrow$

$$\begin{array}{lll} \bar{u} & \langle 3,-9,-2 \rangle, & 3,-9 \\ \times\ \bar{v} & \langle 1,1,1 \rangle, & 1,1 \\ \hline & \langle -7,-5,12 \rangle & \end{array}$$

$\longrightarrow \frac{\|\bar{u} \times \bar{v}\|}{\|\bar{v}\|} = \frac{\sqrt{49+25+144}}{\sqrt{1+1+1}}$

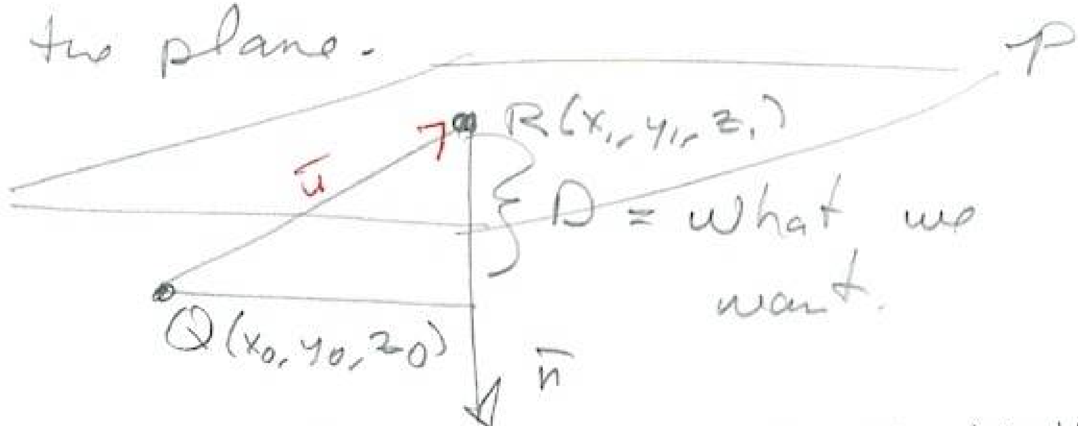
$= \frac{\sqrt{218}}{\sqrt{3}} = \boxed{\sqrt{\frac{218}{3}} = d}$

3c 10 pts Distance from $Q(x_0, y_0, z_0)$ to plane

P: $ax + by + cz = d$ is $\dfrac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2+b^2+c^2}}$

Pf

Let $\bar{n} = \langle a, b, c \rangle$ be normal to the plane. Let $R(x_1, y_1, z_1)$ be any point on the plane.



Let $\bar{u} = \overrightarrow{QR}$. Then $\bar{u} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$

We want $|\mathrm{comp}_{\bar{n}} \bar{u}| = D = \dfrac{|\bar{u} \cdot \bar{n}|}{\|\bar{n}\|}$

$$= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|ax_1 - ax_0 + by_1 - by_0 + cz_1 - cz_0|}{\sqrt{a^2+b^2+c^2}}$$

203 T1

3c Cnt'd

$$= \frac{|ax_1+by_1+cz_1-(ax_0+by_0+cz_0)|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|d-(ax_0+by_0+cz_0)|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$$

3d 10 pts Distance from $Q(-1, 10, 20)$ to

$P: 3x-2y+5z=11$

$$D = \frac{|3(-1)-2(10)+5(20)-11|}{\sqrt{3^2+2^2+5^2}}$$

$$= \frac{|-3-20+100-11|}{\sqrt{9+4+25}} = \boxed{\frac{66}{\sqrt{38}} \text{ OR } \frac{66\sqrt{38}}{38}}$$

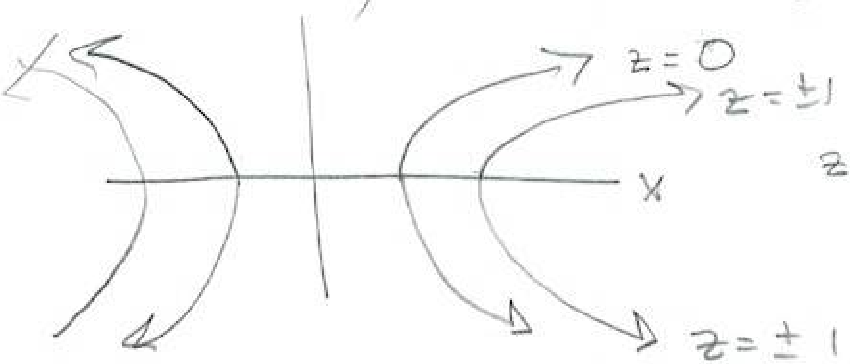
OR $\boxed{\frac{33\sqrt{38}}{19}}$

203 T1

4 10 pts $576x^2 - 324y^2 + 144z^2 = 5184$

$$\to \frac{x^2}{9} - \frac{y^2}{16} + \frac{z^2}{36} = 1$$

xy: $z=0$ $\quad \frac{x^2}{9} - \frac{y^2}{16} = 1$

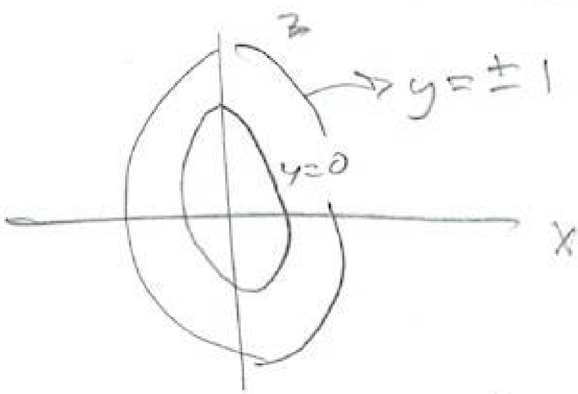


$z = \pm 1$ $\quad \frac{x^2}{9} - \frac{y^2}{16} = 1 - \frac{1}{36} = \frac{35}{36}$

$$\frac{x^2}{9\left(\frac{35}{36}\right)} - \frac{y^2}{16\left(\frac{35}{36}\right)} = 1$$

xz:

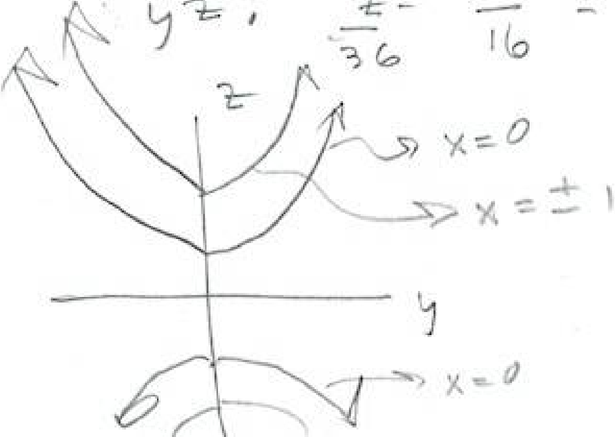
$$\frac{x^2}{9} + \frac{z^2}{36} = \frac{y^2}{16} + 1$$



$y = \pm 1$ $\quad \frac{x^2}{9} + \frac{z^2}{36} = 1 + \frac{1}{16} = \frac{17}{16}$

$$\frac{x^2}{9\left(\frac{17}{16}\right)} + \frac{z^2}{36\left(\frac{17}{16}\right)} = 1$$

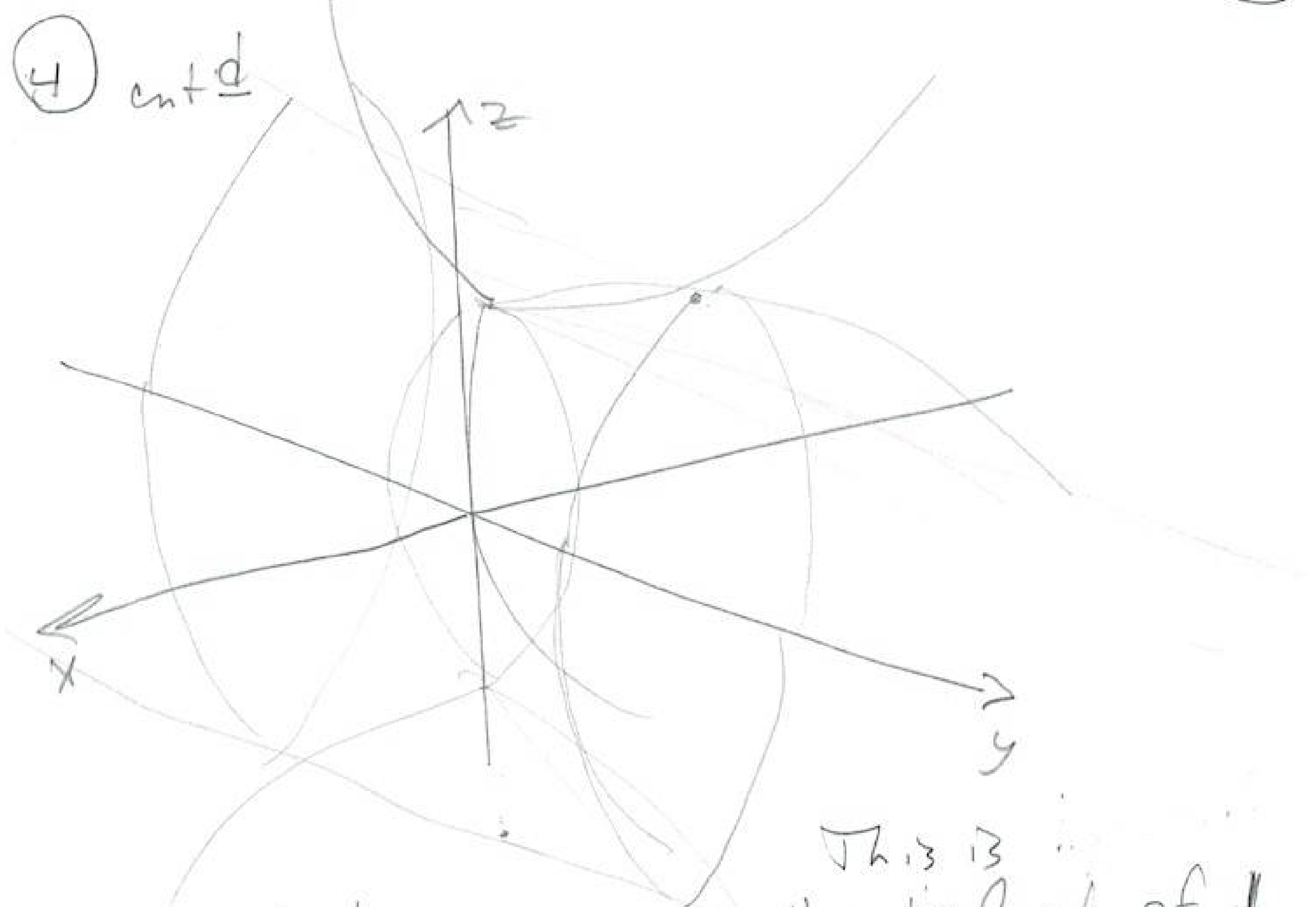
yz: $\frac{z^2}{36} - \frac{y^2}{16} = 1 - \frac{x^2}{9} = \frac{9-x^2}{9}$



203 J1

8

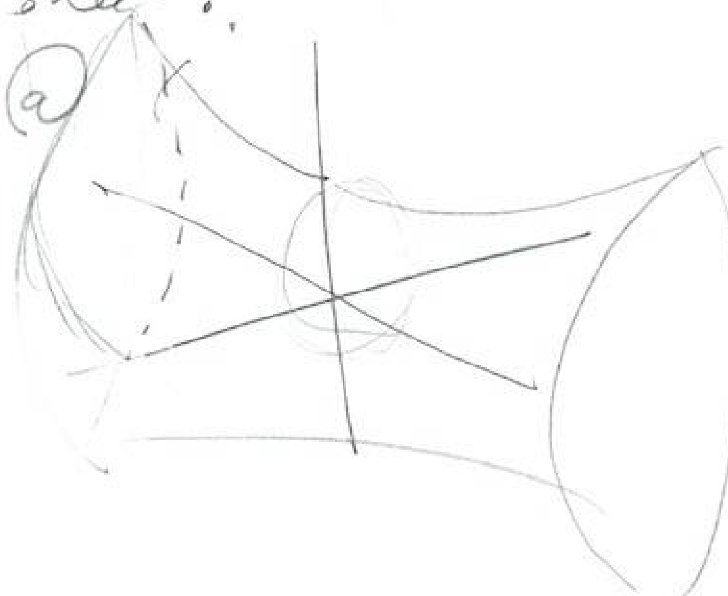
4 cntd



This is Hyperboloid. of II sheet.

Center axis is y-axis

Two cooling towers? is (a)



203   J1   9

5 $\bar{r}(t) = \langle t, \sin(t), \cos(t) \rangle$

a 5pts Find $\bar{T}$ & $\bar{N}$:

$\bar{r}' = \langle 1, \cos(t), -\sin(t) \rangle$

$\|\bar{r}'\| = \sqrt{1+1} = \sqrt{2}$

$\bar{T} = \frac{1}{\sqrt{2}} \bar{r}' = \boxed{\frac{1}{\sqrt{2}} \langle 1, \cos(t), -\sin(t) \rangle = \bar{T}}$

$\bar{N} = \frac{\bar{T}'}{\|\bar{T}'\|} = \frac{\frac{1}{\sqrt{2}} \bar{r}''}{\|\frac{1}{\sqrt{2}} \bar{r}''\|} = \frac{\frac{1}{\sqrt{2}} \langle 0, -\sin t, -\cos t \rangle}{\|\frac{1}{\sqrt{2}} \langle 0, -\sin t, -\cos t \rangle\|}$

$= \frac{\langle 0, -\sin t, -\cos t \rangle}{(\sqrt{1})} = \boxed{\langle 0, -\sin t, -\cos t \rangle = \bar{N}}$

b 5pts $\boxed{\bar{T}\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{2}} \langle 1, \frac{\sqrt{3}}{2}, -\frac{1}{2} \rangle}$ $1, \frac{\sqrt{3}}{2}$

$\boxed{\bar{N}\left(\frac{\pi}{6}\right) = \langle 0, -\frac{1}{2}, -\frac{\sqrt{3}}{2} \rangle}$ $0, -\frac{1}{2}$

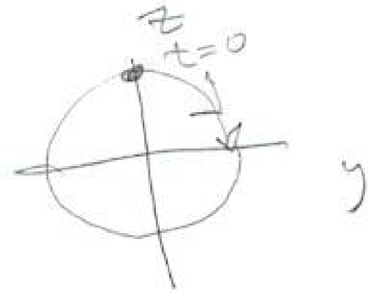
$(\bar{T} \times \bar{N})\left(\frac{\pi}{6}\right) = \bar{B}\left(\frac{\pi}{6}\right)$ $\boxed{= \frac{1}{\sqrt{2}} \langle -1, \frac{\sqrt{3}}{2}, -\frac{1}{2} \rangle = \bar{B}\left(\frac{\pi}{6}\right)}$

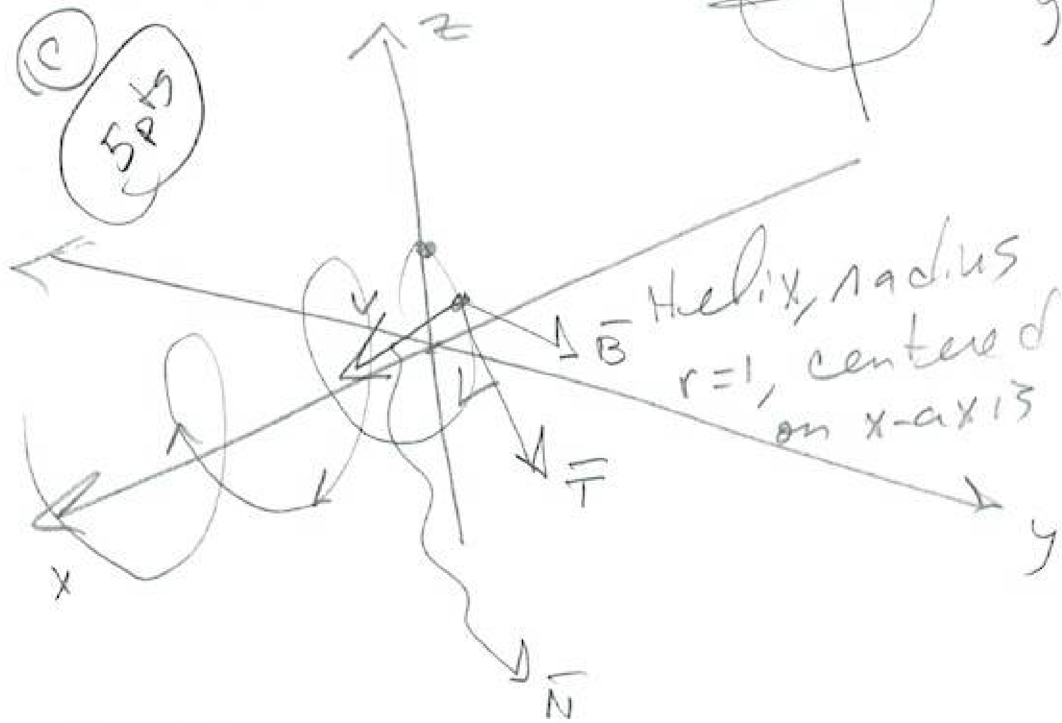
$= \langle -\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{4}, -\frac{\sqrt{2}}{4} \rangle$

203 T1 ⑨

#5 cont'd

(c) 5pts





(6) 5pts $\kappa = \left\| \frac{dT}{ds} \right\|$ is rate of change in direction wrt incremental change in arc length. Literally, the derivative of tangent wrt arc length.

$$\kappa = \frac{\|\bar{r}' \times \bar{r}''\|}{\|\bar{r}'\|^3}$$ is the formula

203 T1 (10)

BONUS SECTION

(7) (5pb) $\bar{r} = \langle 2\sin t, 2\cos t, 1 \rangle$

$\bar{r}' = \langle 2\cos t, -2\sin t, 0 \rangle$, $2\cos t, -2\sin t$

$\bar{r}'' = \langle -2\sin t, -2\cos t, 0 \rangle$, $-2\sin t, -2\cos t$

$\bar{r}' \times \bar{r}'' = \langle 0, 0, -4\cos^2 t - 4\sin^2 t \rangle = 4\langle 0,0,-1\rangle$

$\|\bar{r}' \times \bar{r}''\| = \|4\langle 0,0,-1\rangle\| = 4\|\langle 0,0,-1\rangle\|$

$= 4\sqrt{0+0+1} = 4$

$\|\bar{r}'\| = \sqrt{4} = 2$

$\therefore \quad \kappa = \dfrac{\|\bar{r}' \times \bar{r}''\|}{\|\bar{r}'\|^3} = \dfrac{4}{2^3} = \boxed{\dfrac{1}{2} = \kappa}$

radius of osculating circle is $\frac{1}{\kappa} = \boxed{r = 2}$

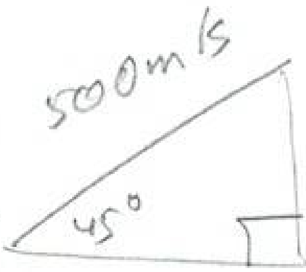
Indeed, $\bar{r}(t)$ is a circle of radius $r = 2$, centered @ $(0,0,1)$ in the plane $z = 1$

203 T1 11

8 5pts $\frac{d}{dx}\left[\int_0^{\sin x} \frac{\tau^5+4\tau^3}{\sqrt{6\tau^3+\cos^2(2t)}}\,d\tau\right]$

$= \left(\frac{\sin^5 x + 4\sin^3 x}{\sqrt{6\sin^3 x + \cos^2(2\sin x)}}\right)(\cos x)$

9 5pts



$x = \left(500\cos\left(\frac{\pi}{4}\right)\right)t$

$y = \left(500\sin\left(\frac{\pi}{4}\right)\right)t - 4.9t^2$

Speed @ impact is 500 m/s

Max Height $\frac{dy}{dt} = -9.8t + (500)\left(\frac{1}{\sqrt{2}}\right) \overset{SET}{=} 0$

$\rightarrow 9.8t - 250\sqrt{2} = 0$ $\boxed{y(t_0) \approx 6377.551024}$

$9.8t = 250\sqrt{2}$

$36.07687654 \approx t_0 = \frac{250\sqrt{2}}{9.8} = \frac{2500\sqrt{2}}{98} = \frac{1250\sqrt{2}}{49}$ sec

MAX. HT:

$y\left(\frac{1250\sqrt{2}}{49}\right) = (250\sqrt{2})\left(\frac{1250\sqrt{2}}{49}\right) - 4.9\left(\frac{1250\sqrt{2}}{49}\right)^2$

DISTANCE DOWN RANGE:

$x\left(\frac{1250\sqrt{2}}{49}\right) = (250\sqrt{2})\left(\frac{1250\sqrt{2}}{49}\right) \approx 12{,}755.10204$ m

TIMES 2 ≈ 25,510.20408

$y'\left(\frac{250\sqrt{2}}{9.8}\right) = -353.5533906 = -y'(0)\ \frac{m}{s}$

$\left.\frac{dx}{dt}\right|_{t=0} = \frac{500}{\sqrt{2}} \approx 353.5533906\ \frac{m}{s}$

203 T1 12

10 5pts $\vec{r}(t), \vec{v}(t)$ from $\vec{a}(t) = \langle \frac{1}{t^2+1}, \sec t, t \rangle$

$\vec{v}(t) = \langle \arctan(x), \ln(\sec x + \tan x), \frac{1}{2}t^2 \rangle + \vec{C_1}$

$\vec{v}(0) = \langle 2, 3, 1 \rangle = \langle \arctan(0),$