203 §16.7 #s 7, 9, 14, 17, 24, 35, 41

~~7~~ #s 5-18 Evaluate the surface integral

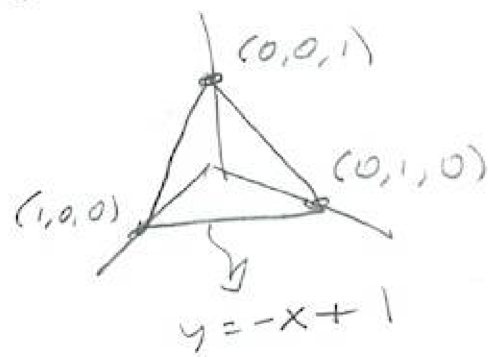
7. $\iint_S yz\,dS$, S: the part of $x+y+z=1$ that lives in the 1st octant.

$z = 1-x-y$, $\bar{r} = \langle x, y, 1-x-y \rangle$

$$\begin{array}{rl} & \bar{r}_x = \langle 1, 0, -1 \rangle, 1, 0 \\ \times & \bar{r}_y = \langle 0, 1, -1 \rangle, 0, 1 \\ \hline & \langle 1, 1, 1 \rangle \end{array}$$

$\langle 1,1,1\rangle$ & $\|\bar{r}_x \times \bar{r}_y\| = \sqrt{3}$

Projection into xy-plane:



$(1,0), (0,1)$

$m = \frac{1-0}{0-1} = -1$

$y = -(x-1) + 0$

$= -x+1$

$$\therefore \iint_S yz\,dS = \int_0^1 \int_0^{1-x} \underbrace{y(1-x-y)}_{\to (1-x)y - y^2} \sqrt{3}\,dy\,dx$$

$$= \int_0^1 \sqrt{3} \int_0^{1-x} \underbrace{(y - xy - y^2)}_{(1-x)y-y^2}\,dy\,dx = \int_0^1 \sqrt{3}\left[(1-x)\left(\frac{y^2}{2}\right) - \frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \sqrt{3}\int_0^1 \left[(1-x)\left(\frac{(1-x)^2}{2}\right) - (1-x)^3\right] dx = \sqrt{3}\int_0^1 -\frac{1}{2}(1-x)^3\,dx$$

$$= \frac{\sqrt{3}}{2}\left.\frac{\left[(1-x)^4\right]}{4}\right]_0^1 = \frac{\sqrt{3}}{2}\left[\frac{1}{4}\right] = \frac{\sqrt{3}}{8}$$

203 §16.7 # s 14, 17, 24, 35, 41

(14) $\iint_S y^2\,dS$, S: part of sphere $x^2+y^2+z^2=4$

that lies inside the cylinder $x^2+y^2=1$

& above the xy-plane.

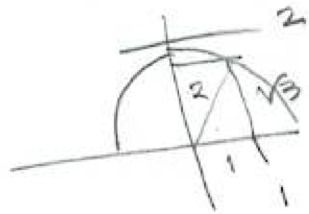
$y=0$:

$x^2+z^2=4$

$x^2=1$

$x=\pm 1$

$2^2-1^2=3$



$0\le\phi\le\frac{\pi}{6}$

$0\le\theta\le 2\pi$

$x=2\sin\phi\cos\theta$, $y=2\sin\phi\sin\theta$, $z=2\cos\phi$

$\iint_S y^2\,dS$

$\bar r = \langle 2\sin\phi\cos\theta,\ 2\sin\phi\sin\theta,\ 2\cos\phi\rangle$

$\langle 2\cos\phi\cos\theta,\ 2\cos\phi\sin\theta,\ -2\sin\phi\rangle$,

$\bar r_\phi = \langle 2\cos\phi\cos\theta,\ 2\cos\phi\sin\theta,\ -2\sin\phi\rangle$, $2\cos\phi\cos\theta$, $2\cos\phi\sin\theta$

$\bar r_\phi = \langle 2\cos\phi\cos\theta,\ 2\cos\phi\sin\theta,\ 0\rangle$, $-2\sin\phi\sin\theta$, $2\sin\phi\cos\theta$

$\times\ \bar r_\theta = \langle -2\sin\phi\sin\theta,\ 2\sin\phi\cos\theta,\ 4\sin\phi\cos\phi\cos^2\theta + 4\sin\phi\cos\phi\sin^2\theta$

$\langle 4\sin^2\phi\cos\theta,\ 4\sin^2\phi\sin\theta,\ 4\sin\phi\cos\phi\rangle = \bar r_\phi\times\bar r_\theta$

$= \langle 4\sin^2\phi\cos\theta,\ 4\sin^2\phi\sin\theta,$

$\Rightarrow \|\bar r_\phi\times\bar r_\theta\| = (16\sin^4\phi\cos^2\theta + 16\sin^4\phi\sin^2\theta + 16\sin^2\phi\cos^2\phi)^{\frac{1}{2}}$

$= 4(\sin^4\phi + \sin^2\phi\cos^2\phi)^{\frac{1}{2}} = 4\sqrt{\sin^2\phi(\sin^2\phi+\cos^2\phi)}$

$= 4\sin\phi \qquad 0\le\phi\le\pi$

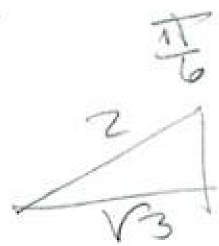
203 $\quad$ § 16.7 #s 14, 17, 24, 35, 41

#14 cont'd.

This gives

$$\iint_S y^2\,dS = \int_0^{2\pi}\int_0^{\frac{\pi}{6}} (2\sin\phi\sin\theta)^2 (4\sin\phi)\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{6}} (4\sin^2\phi\sin^2\theta)(4\sin\phi)\,d\phi\,d\theta$$



$$= 16\int_0^{2\pi}\sin^2\theta\,d\theta\int_0^{\frac{\pi}{6}}\sin^3\phi\,d\phi$$

$$= \left(16\int_0^{2\pi}\frac{1}{2}(1+\cos(2\theta))\,d\theta\right)\int_0^{1}(1-\cos^2\phi)\sin\phi\,d\phi$$

$$= 8\left[\theta+\frac{1}{2}\sin(2\theta)\right]_0^{2\pi}\left[-\cos\phi+\frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{6}}$$

$$= 8[2\pi]\left[-\cos\frac{\pi}{6}+\frac{\cos^3\frac{\pi}{6}}{3}-\left(-\cos 0+\frac{\cos^3 0}{3}\right)\right]$$

$$= 16\pi\left[-\frac{\sqrt{3}}{2}+\frac{\left(\frac{\sqrt{3}}{2}\right)^3}{3}+1-\frac{1}{3}\right] = 16\pi\left[-\frac{\sqrt{3}}{2}+\frac{3\sqrt{3}}{8\cdot 3}+1-\frac{1}{3}\right]$$

$$= 16\pi\left[\frac{-4\sqrt{3}+\sqrt{3}}{8}+\frac{2}{3} = \frac{-3\sqrt{3}}{8}+\frac{2}{3}\right.$$

$$= \boxed{\frac{-9\sqrt{3}+16}{24}}$$

203 Skirt #s 9, 14, 17, 24, 35, 41

(9) $\iint_S yz\,dS$ S: $x = u^2$, $y = u\sin v$, $z = u\cos v$

$0 \le u \le 1$, $0 \le v \le \frac{\pi}{2}$

$$\begin{array}{lll} & \bar{r}_u = \langle 2u, \sin v, \cos v \rangle & , 2u, \sin v \\ \times & \bar{r}_v = \langle 0, u\cos v, -u\sin v, & 0, u\cos v \end{array}$$

$\langle -u\sin^2 v - u\cos^2 v, 2u^2\sin v, 2u^2\cos v \rangle$

$= \langle -u, 2u^2\sin v, 2u^2\cos v \rangle$

$\|\bar{r}_u \times \bar{r}_v\| = \sqrt{u^2 + 4u^4\sin^2 v + 4u^4\cos^2 v}$

$= \sqrt{u^2 + 4u^4} = u\sqrt{1 + 4u^2}$

This gives

$$\int_0^1 \int_0^{\frac{\pi}{2}} (u\sin v)(u\cos v)\, u\sqrt{4u^2+1}\, dv\, du$$

$$= \int_0^1 u^3\sqrt{4u^2+1}\, du \int_0^{\frac{\pi}{2}} \sin v \cos v\, dv$$

$u = u^2$ $dv = \sqrt{4u^2+1}\, 8u\, du \cdot \frac{1}{8}$

$du = 2u\, du$ $v = \frac{2}{3}(4u^2+1)^{\frac{3}{2}} \cdot \frac{1}{8}$

$uv - \int v\, du = \frac{2}{3}u^2(4u^2+1)^{\frac{3}{2}} - \int 2u \cdot \frac{2}{3}(4u^2+1)^{\frac{3}{2}}\, du$

$(8u\, du)\Big]_0 =$

$= \frac{\frac{2}{3}u^2(4u^2+1)^{\frac{3}{2}}}{8} - \frac{4}{3}\cdot\frac{1}{8}\int (4u^2+1)^{3/2}(8u\, du)$

$\frac{\frac{2}{3}}{8} - \frac{1}{12}\left[\frac{(4u^2+1)^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^1 = \frac{1}{12} - \frac{1}{12}\left[\frac{5^{\frac{5}{2}}}{\frac{5}{2}} - \frac{2}{5}\right]$

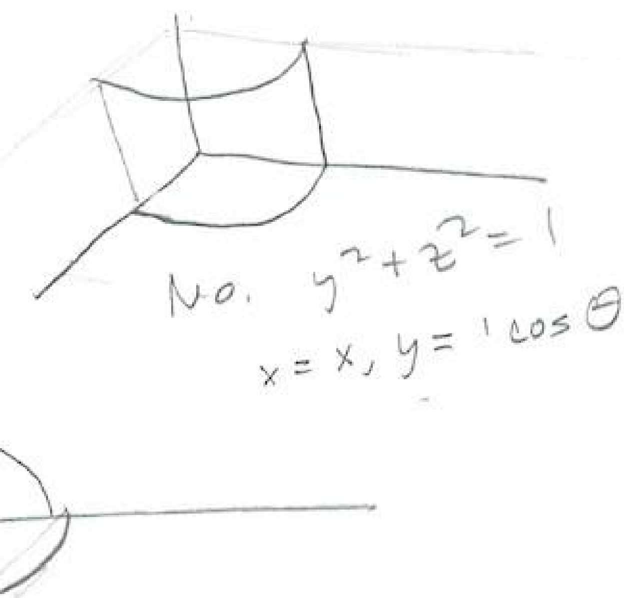
$= \frac{1}{12} - \frac{1}{12}\left[\frac{2}{5}(25\sqrt{5}) - \frac{2}{5}\right]$

Wow That #9 was messy!

203 § 16.7

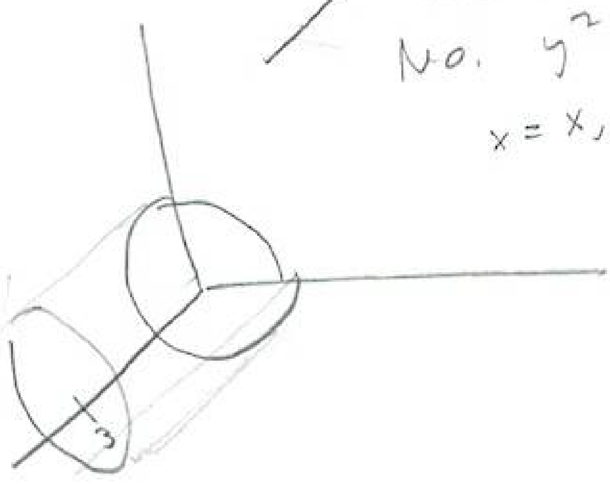
Use Maple to eval some of these, after you've set them up.

(17) $\iint_S (z + x^2 y)\, dS$ S: part of cylinder $x^2+y^2=1$ that lies between $x=0$ & $x=3$ in Oct I.



No. $y^2+z^2=1$

$x = x$, $y = 1\cos\theta$, $z = \sin\theta$



203 §16.7 #s 17, 24, 35, 41

#17 cnt'd

$\bar{r}_x = \langle 1, 0, 0 \rangle$, 1, 0

$r_\theta = \langle 0, -\sin\theta, \cos\theta \rangle$, 0, $-\sin\theta$

$\langle 0, -\cos\theta, -\sin\theta \rangle \Rightarrow \|\bar{r}_x \times \bar{r}_\theta\| = 1$

$$\Rightarrow \int_0^3 \int_0^{2\pi} (\sin\theta + x^2\cos\theta)\, d\theta\, dx$$

$$= \int_0^3 \Big[-\cos\theta + x^2\sin\theta\Big]_0^{2\pi} dx$$

$$= \int_0^3 (-\cos 2\pi - (-\cos 0) + x^2(0-0))\, dx$$

$$= \int_0^3 0\, dx = 0$$

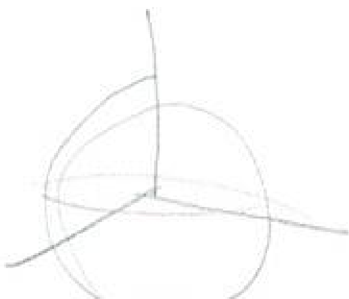
203 §16.7 #s 24, 35, 41

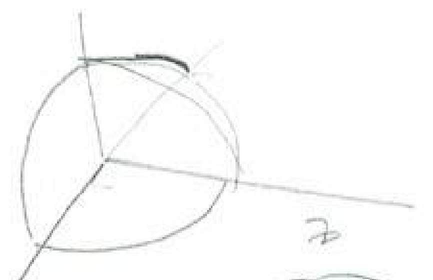
~~24~~ #s 21–32 Eval surface integral

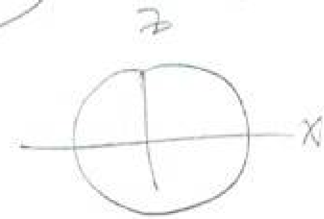
$\iint_S \vec{F} \cdot d\vec{S}$ for $\vec{F}$ & oriented surface S

Find flux, i.e.

24 $\vec{F} = \langle xz, x, y \rangle$, S: $x^2+y^2+z^2=25$, $y \geq 0$
oriented in direction of pos. y-axis







$\langle x, 1-x^2-z^2, z \rangle$

$\vec{r}_x = \langle 1, -2x, 0 \rangle$, 1, −2x

$\vec{r}_z = \langle 0, -2z, 1 \rangle$, 0, −2z

$\langle -2x, -1, -2z \rangle \Rightarrow \|\vec{r}_x \times \vec{r}_z\| = \sqrt{4x^2+4z^2+1}$

$\vec{F} \cdot (\vec{r}_x \times \vec{r}_z)\, dA = (-2x^2 z - x - 2yz)\, dz\, dx$

$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(-x^2 z^2 - xz - yz^2\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = \int_{-1}^{1}$

203 S'16.7 #s 24, 35, 41

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \qquad y=\sqrt{1-x^2-z^2}$$

$$\vec{F}\cdot(\vec{r}_x \times \vec{r}_z) = \langle xz, x, y\rangle \cdot \langle -2x, -1, -2z\rangle$$

$$= -2x^2z - x - 2yz$$

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (-2x^2z - x - 2yz)\,dz\,dx$$

$$\cancel{= \int_{-1}^{1}\left[-x^2z^2 - xz - yz^2\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx}$$

$$\cancel{= \int_{-1}^{1} -x^2(1-x^2) + x\sqrt{1-x^2} -}$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(-2x^2z - x - 2\sqrt{1-x^2-z^2}\, z\right) dz\,dx$$

$$+ (1-x^2-z^2)^{\frac{1}{2}}(-2z)\,dz$$

$$= \int_{-1}^{1}\left[-x^2z^2 - xz + \frac{2}{3}(1-x^2-z^2)^{\frac{3}{2}}\right]_{z=-\sqrt{1-x^2}}^{z=\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1}\Big[-x^2(1-x^2) - x\sqrt{1-x^2} + \frac{2}{3}\left(\overset{0}{1-x^2-(1-x^2)}\right)^{3/2}$$

$$+ \frac{2}{3}(0)\Big]\,dx$$

$$- \left[-x^2(1-x^2) - x(-\sqrt{1-x^2})\right.$$

$$= 0?$$

203 § 16.7 #s 24, 35, 41

24 cont'd

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$0 \le \phi \le \pi$$

$$\rho = 1$$

$$\bar{r}_\phi \times \bar{r}_\theta = \rho^2 \sin\phi = \sin\phi$$

$$x = \sin\phi\cos\theta,\ y = \sin\phi\sin\theta,\ z = \cos\phi$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi}$$

$$\bar{F} \cdot \bar{r}_\phi \times \bar{r}_\theta$$

$$= \bar{F} = \langle \sin\phi\cos\theta\cos\phi,\ \sin\phi\cos\theta,\ \sin\phi\sin\theta \rangle$$

$$\langle \sin^2\phi\cos\theta,\ \sin^2\phi\sin\theta,\ 4\sin\phi\cos\theta \rangle$$

$$\cdot \langle \sin^2\phi\cos\theta,\ \sin^2\phi\sin\theta,$$

$$= 4\sin^3\phi\cos\phi\cos^2\theta + \sin^3\phi\cos\theta\sin\theta + \sin^2\phi\sin\theta\cos\theta$$

This gives

$$\int_0^{\pi} \sin^3\phi\cos\phi\, d\phi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1+\cos(2\theta))\, d\theta$$

$$+ \int_0^{\pi} (1-\cos^2\phi)\sin\phi\, d\phi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\cos\theta\, d\theta$$

$$+ \int_0^{\pi} \frac{1}{2}(1-\cos(2\phi))\, d\phi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\cos\theta\, d\theta$$

203 §16.7 #24 cont'd

$$= \frac{1}{4}\sin^4\phi\Big]_0^{\pi} \, \frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$+ \left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_0^{\pi}\left[\frac{1}{2}\sin^2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$+ \left[\frac{1}{2}\phi - \frac{1}{4}\sin(2\phi)\right]_0^{\pi}\left[\frac{1}{2}\sin^2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= [0][\sim] + [\sim][0] + [\sim][0] = 0$$

203 §16.7 #s 35, 41

(35) Find formula similar to $\boxed{10}$

for the case when $y = h(x, z)$ and $\bar{n}$ is the unit normal that points towards the left.

$$\bar{r} = \langle x, h(x,z), z \rangle$$

$$\bar{r}_x = \langle 1, h_x, 0 \rangle, 1, h_x$$

$$\times \quad \bar{r}_z = \langle 0, h_z, 1 \rangle, 0, h_z$$

$$\langle h_x, -1, h_z \rangle$$ for positive orientation

so the $\bar{n}$ we want is $-\bar{n}$!

$$-\bar{F} \cdot \bar{r}_x \times \bar{r}_z = \langle P, Q, R \rangle \cdot \langle -h_x, +1, -h_z \rangle$$

$$= -P h_x + Q - R h_z$$

$$\iint_S \bar{F} \cdot d\bar{S} = \iint_D (-P h_x + Q - R h_z)\, dA$$

(41) Fluid density $\rho = 870 \frac{kg}{m^3}$

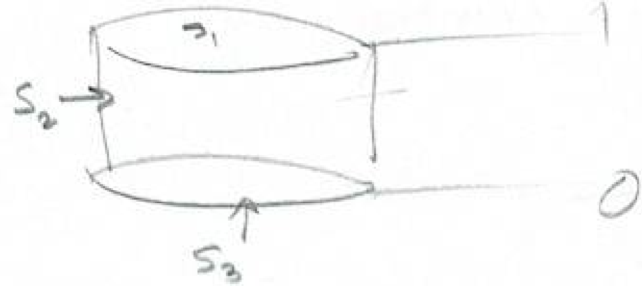
velocity $= \bar{v} = \langle z, y^2, x^2 \rangle$ $x, y, z \in$ meters per second

Find rate of flow through the cylinder

$x^2 + y^2 = 4, \quad 0 \le z \le 1$

203 S16.7 #s 41

41 cnt'd



S_1 : $\bar{n}_1 = \langle 0, 0, 1 \rangle$

S_3 : $\bar{n}_3 = \langle 0, 0, -1 \rangle$

S_2 : $\bar{r} = \langle 2\cos\theta, 2\sin\theta, z \rangle$

$\bar{r}_\theta = \langle -2\sin\theta, 2\cos\theta, 0 \rangle, -2\sin\theta, 2\cos\theta$

$\times\ \bar{r}_z = \langle 0, 0, 1 \rangle, 0, 0$

$\langle 2\cos\theta, 2\sin\theta, 0 \rangle = \bar{r}_\theta \times \bar{r}_z$

~~$\bar{r}_\theta \times$~~ is outward normal that we want.

~~$\iint \bar{F} \cdot d\bar{S} = \int$~~

S_1 : $0 \le r \le 2$, $0 \le \theta \le 2\pi$, $z = 1$

S_3 : $0 \le r \le 2$, $0 \le \theta \le 2\pi$, $z = 0$

$\bar{F} \cdot \bar{n}_1 = x^2 = 4\cos^2\theta$ $\quad \int_0^2 \int_0^{2\pi} 4\cos^2\theta \, d\theta \, dr$

$\bar{F} \cdot n_2 = -x^2 = -4\cos^2\theta$ $\quad + \int_0^2 \int_0^{2\pi} -4\cos^2\theta \, d\theta \, dr$

203 §16.7 #41 cont'd

41 cont'd So, it appears the thing comes down to the side wall.

$$\bar{r}_\theta \times \bar{r}_z = \langle 2\cos\theta, 2\sin\theta, 0\rangle$$

$$\bar{F} \cdot \bar{r}_\theta \times \bar{r}_z = \langle z, 4\sin^2\theta, 4\cos^2\theta\rangle \cdot \langle 2\cos\theta, 2\sin\theta, 0\rangle$$

$$= 2z\cos\theta + 8\sin^3\theta$$

$$\int_0^{2\pi}\int_0^1 \left(2z\cos\theta + 8\sin\theta(1-\cos^2\theta)\right)dz\,d\theta$$

$$= \int_0^{2\pi}\left[z^2\cos\theta + 8z\sin\theta - 8z\cos^2\theta\sin\theta\right]_0^1 d\theta$$

$$= \int_0^{2\pi}(\cos\theta + 8\sin\theta - 8\cos^2\theta\sin\theta)\,d\theta$$

$$= \left[\sin\theta - 8\cos\theta + \frac{8}{3}\cos^3\theta\right]_0^{2\pi}$$

$$= 0 - 8 + \frac{8}{3} - \left(0 - 8 + \frac{8}{3}\right)$$

$$= 0.$$