203 §16.5 #s 1, 3, 7, 13, 17, 20, 21, 22, (23-29, 38)

#s 1-8 (a) Curl (b) Divergence

1) $\bar{F} = \langle xyz, 0, -x^2y \rangle$

a) $\nabla \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$

$\times \quad \langle xyz, 0, -x^2y \rangle, xyz, 0$

$\nabla \times \bar{F} = \langle -x^2, xy - 2xy, -xz \rangle$

$= \boxed{\langle -x^2, -xy, -xz \rangle = \text{curl } \bar{F}}$

b) $\nabla \cdot \bar{F} = yz + 0 - x^2 = \boxed{yz - x^2 = \text{div } \bar{F}}$

3) $\bar{F} = \langle$

203 §16.5 #s ~~13-18~~ 13, 17, 20, 21, 22 (23-29, 30)

#s 13-18 Is $\bar{F}$ conservative? If yes, then find $f \ni \nabla f = \bar{F}$.

(13) $\bar{F} = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$, $2xyz^3$, $3xy^2z^2$

$\nabla \times \bar{F} = \langle 6xyz^2 - 6xyz^2, 6xyz^2 - 3y^2z^2, 3y^2z^2 - 2xz^3 \rangle$

$\neq \bar{0} \Longrightarrow$ not conservative.

bc $\bar{F}$ is cnt^{s}ly difbl on $\mathbb{R}^3$.

(17) $\bar{F} = \langle ye^{-x}, e^{-x}, 2z \rangle$, ye^{-x}, e^{-x}

$\nabla \times \bar{F} = \langle 0, 0, -e^{-x} - e^{-x} \rangle = \langle 0, 0, -2e^{-x} \rangle \neq \bar{0}$

~~$\Longrightarrow f_x = ye^{-x} \Longrightarrow f = -ye^{-x} + g(y,z) \Longrightarrow f_y = -e^{-x} + g_y(y,z)$~~

~~$f_y = e^{-x}$~~

~~$f_z = 2z$~~ Not cnsrvtv.

(20) Does $\exists\ \bar{G}$ on $\mathbb{R}^3 \ni \text{curl}\,\bar{G} = \langle xyz, -y^2z, yz^2 \rangle$?

Let's see:

$\text{div}(\text{curl}\,\bar{G}) = \text{div}(\langle xyz, -y^2z, yz^2 \rangle)$

$= yz - 2yz + yz = 0 \Longrightarrow$ Yes.

This is Thm 11, b/c $\bar{F}$'s entries are ∞'ly cnt^{s}ly difbl, and all we need are twice-cnt^{s}ly difbl

209 §16.5 #s 21, 22 (23-29, 30)

(21) Show that any vector field of the form $\overline{F} = \langle f(x), g(y), h(z) \rangle$ is <u>irrotational</u>.

$\operatorname{curl} \overline{F} = \langle h_y(z) - g_z(y), f_z(x) - h_x(z), g_x(y) - f_y(x) \rangle$

$\langle f(x), g(y), h(z) \rangle, f(x), g(y)$

$= \overline{0} \Longrightarrow$ cnsrvtv $\Longrightarrow$ No tendency to rotate.

(22) $\overline{F} = \langle f(y,z), g(x,z), h(x,y) \rangle$ is incompressible.

$\operatorname{div} \overline{F} = f_x(y,z) + g_y(x,z) + h_z(x,y) = 0$

$\Longrightarrow$ No divergence, i.e., incompressible.

(23) $\overline{F} = \langle f_1, f_2, f_3 \rangle$, $\overline{G} = \langle g_1, g_2, g_3 \rangle$

$\Longrightarrow \operatorname{div}(\overline{F} + \overline{G}) = \nabla \cdot (\overline{F} + \overline{G})$

$= \nabla \cdot \langle f_1 + g_1, f_2 + g_2, f_3 + g_3 \rangle$

$= (f_1 + g_1)_x + (f_2 + g_2)_y + (f_3 + g_3)_z$

$= f_{1x} + g_{1x} + f_{2y} + g_{2y} + f_{3z} + g_{3z}$

because $\frac{\partial}{\partial x}$ is linear, so $\frac{\partial}{\partial x}(f_1 + g_1) = \frac{\partial f_1}{\partial x} + \frac{\partial g_1}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$ the same, so rearranging &

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#23 cnt'd

$$(f_{1_x} + f_{2_y} + f_{3_z}) + (g_{1_x} + g_{2_y} + g_{3_3})$$

$$= \nabla \cdot \bar{F} + \nabla \cdot \bar{G}.$$

We also used commutativity and associativity of addition.

(24) curl $(\bar{F} + \bar{G})$ = curl $\bar{F}$ + curl $\bar{G}$

$\bar{F} = \langle P, Q, R \rangle$, $\bar{G} = \langle S, T, U \rangle$.

Again, the linearity of the differential operators $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial z}$ & $\frac{\partial}{\partial y}$ will carry the day.

$$\bar{F} + \bar{G} = \langle \overset{x}{P+S}, \overset{y}{Q+T}, \overset{z}{R+U} \rangle, \overset{x}{P+S}, \overset{y}{Q+T}$$

$$\nabla \times (\bar{F} + \bar{G}) = \langle (R+U)_y - (Q+T)_z, (P+S)_z - (R+U)_x, (Q+T)_x - (P+S)_y \rangle$$

$$= \langle R_y + U_y - Q_z - T_z, P_z + S_z - R_x - U_x, Q_x + T_x - P_y - S_y \rangle$$

$$= \langle R_y - Q_z + U_y - T_z, P_z - R_x + S_z - U_x, Q_x - P_y + T_x - S_y \rangle$$

$$= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle + \langle U_y - T_z, S_z - U_x, T_x - S_y \rangle$$

$$= \text{curl } \bar{F} + \text{curl } \bar{G}$$

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25 $\text{div}(f\bar{F}) = f\,\text{div}\bar{F} + \bar{F}\cdot\nabla f$

$\text{div}(f\bar{F}) = \text{div}(\langle fP, fQ, fR)$

$= \frac{\partial}{\partial x}(fP) + \frac{\partial}{\partial y}(fQ) + \frac{\partial}{\partial z}(fR)$

$= f_xP + fP_x + f_yQ + fQ_y + f_zR + fR_z$

$= f_xP + f_yQ + f_zR + fP_x + fQ_y + fR_z$

$= \nabla f\cdot\bar{F} + f\,\nabla\cdot\bar{F}$

$= \bar{F}\cdot\nabla f + f\,\text{div}\,\bar{F}$, by commutativity of dot product & defin of $\text{div}\,\bar{F}$.

26 $\text{curl}(f\bar{F}) = f\,\text{curl}\,\bar{F} + \nabla f\times\bar{F}$

$= f\,\nabla\times\bar{F} + \nabla f\times\bar{F}$

$\text{curl}(f\bar{F})$: $\langle \overset{x}{fP}, \overset{y}{fQ}, \overset{z}{fR}\rangle, \overset{x}{fP}, \overset{y}{fQ}$

$\text{curl}(f\bar{F}) = \langle (fR)_y - (fQ)_z, (fP)_z - (fR)_x, (fQ)_x - (fP)_y\rangle$

$= \langle f_yR + fR_y - f_zQ - fQ_z,\ f_zP + fP_z - f_xR - fR_x,$
$f_xQ + fQ_x - f_yP - fP_y\rangle$

$= \langle f_yR - f_zQ + fR_y - fQ_z,\ f_zP - f_xR + fP_z - fR_x,$
$f_xQ - f_yP + fQ_x - fP_y\rangle$

$= \nabla f\times\bar{F} + f\,\nabla\times\bar{F} = \nabla f\times\bar{F} + f\,\text{curl}\,\bar{F} =$
$f\,\text{curl}\,\bar{F} + \nabla f\times\bar{F}$ ∎

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~~#26 ...~~

27 $\operatorname{div}(\bar F \times \bar G) = \bar G \circ \operatorname{curl} \bar F - \bar F \circ \operatorname{curl} \bar G$

$= \bar G \circ \nabla \times \bar F - \bar F \circ (\nabla \times \bar G)$

$$\begin{array}{r} \langle P, Q, R \rangle, P, Q \\ \times\ \langle S, T, U \rangle, S, T \\ \hline \langle QU - RT,\ RS - PU,\ PT - QS \rangle = \bar F \times \bar G \end{array}$$

$\rightarrow \operatorname{div}(\bar F \times \bar G) =$

$Q_x U + Q U_x - R_x T - R T_x + R_y S + R S_y - P_y U - P U_y$

$+ P_z T + P T_z - Q_z S - Q S_z$

$= \cancel{Q_x U - R_x T + Q U_x - R T_x + R_y S - P_y U} + R S_y - P U_y$

$\cancel{+ P_z T - Q_z S + P T_z} - Q S_z$

$= Q U_x - R T_x + R S_y - P U_y + P T_z - Q S_z$

$+ Q_x U - R_x T + R_y S - P_y U + P_z T - Q_z S$

$= P(T_z - U_y) + Q(U_x - S_z) + R(S_y - T_x)$

$+ U(Q_x - P_y) + S(R_y - Q_z) + T(P_z - R_x)$

$= \bar F \circ \langle T_z - U_y, U_x - S_z, S_y - T_x \rangle + \bar G \circ \langle Q_x - P_y, R_y - Q_z, P_z - R_x \rangle$

$$\begin{array}{cccccc} & x & y & z & x & y \\ x & \langle P, & Q, & R\rangle, & P, & Q \end{array}$$

$$\text{curl } \bar{F} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$$

$$\begin{array}{ccccc} \langle x, & y, & z\rangle, & x, & y \\ \langle S, & T, & U\rangle, & S, & T \end{array}$$

$$\text{curl } \bar{G} = \langle U_y - T_z,\ S_z - U_x,\ T_x - S_y \rangle$$

So, yes, the negative of $\bar{F} \cdot \text{curl } \bar{G}$ in the sum on the right hand side.

We have

$$\bar{F} \cdot \langle T_z - U_y,\ U_x - S_z,\ S_y - T_x\rangle + \bar{G} \cdot \langle Q_x - P_y,\ R_y - Q_z,\ P_z - R_x\rangle.$$

See *

$$= -\bar{F} \cdot \text{curl } \bar{G} + \bar{G} \cdot \text{curl } \bar{F}$$

(28) $\text{div}(\nabla f \times \nabla g) = 0$

By #27, we have $\text{div}(\bar{F} \times \bar{G}) = \bar{G} \cdot \text{curl } \bar{F} - \bar{F} \cdot \text{curl } \bar{G}$

$$= (\nabla g) \cdot \text{curl}(\nabla f) - (\nabla f) \cdot \text{curl}(\nabla g)$$

$$= \nabla g \cdot \bar{0} + \nabla f \cdot \bar{0} = 0.$$

203 §16.5 You guys deserve a better treatment of §16.5 ~~#s~~ 29, 38.

(29) $\operatorname{curl}(\operatorname{curl} \bar{F}) = \operatorname{grad}(\operatorname{div} \bar{F}) - \nabla^2 \bar{F}$

Proof

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$$

$$\langle P, Q, R \rangle \; P, Q$$

$$\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$\begin{array}{ccccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \end{array}$$

$$\times \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \; R_y - Q_z, P_z - R_x$$

$$\langle Q_{xy} - P_{yy} - P_{zz} + R_{xz}, R_{yz} - Q_{zz} - Q_{xx} + P_{yx},$$

$$P_{zx} - R_{xx} - R_{yy} + Q_{zy} \rangle = \operatorname{curl}(\operatorname{curl} \bar{F})$$

DEF'N OF $\nabla^2 \bar{E}$

$$\operatorname{grad}(\operatorname{div} \bar{F}) - \nabla^2 \bar{F} =$$

$$\nabla(P_x + Q_y + R_z) - \langle \nabla^2 P, \nabla^2 Q, \nabla^2 R \rangle$$

$$= \langle P_{xx} + Q_{yx} + R_{zx}, P_{xy} + Q_{yy} + R_{zy}, P_{xz} + Q_{yz} + R_{zz} \rangle$$

$$- \langle P_{xx} + P_{yy} + P_{zz}, Q_{xx} + Q_{yy} + Q_{zz}, R_{xx} + R_{yy} + R_{zz} \rangle$$

$$= \langle Q_{yx} + R_{zx} - P_{yy} - P_{zz}, P_{xy} + R_{zy} - Q_{xx} - Q_{zz},$$

$$P_{xz} + Q_{yz} - R_{xx} - R_{yy} \rangle \; \blacksquare$$

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Now #38 follows instantly, if you can get past $\nabla \times \frac{\partial}{\partial t} \bar{E} = \frac{\partial}{\partial t} \nabla \times \bar{E}$

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which poses no real hardship, although I'm still a bit confused about this partial w.r.t business, when everything in sight is in $\mathbb{R}^3$ $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

(a) $\nabla \times (\nabla \times \bar{E}) = \nabla \times \left(-\frac{1}{c} \frac{\partial \bar{H}}{\partial t}\right)$

$= -\frac{1}{c} \frac{\partial}{\partial t}(\nabla \times \bar{H}) = \frac{\partial}{\partial t}\left(-\frac{1}{c}\left(\frac{1}{c} \frac{\partial \bar{E}}{\partial t}\right)\right)$

$= -\frac{1}{c^2} \frac{\partial^2 \bar{E}}{\partial t^2}.$

(b) Same deal as (a), just flip $\bar{E}$ & $\bar{H}$

(c) $\nabla^2 E = \text{grad}(\text{div } \bar{E}) - \text{curl}(\text{curl}(\bar{E}))$

$= \text{curl}(\text{curl } \bar{E}) =$

By #29 $= 0 - \text{curl}(\text{curl } \bar{E})$

$\text{curl}(\text{curl } \bar{F}) = \text{grad}(\text{div } \bar{E}) - \nabla^2 \bar{E}$

$= -\left(-\frac{1}{c^2} \frac{\partial^2 \bar{E}}{\partial t^2}\right) = \frac{1}{c^2} \frac{\partial^2 \bar{E}}{\partial t^2}$, by (a)

(d) Same deal, but switch $\bar{E}$ & $\bar{H}$ (& change the sign)