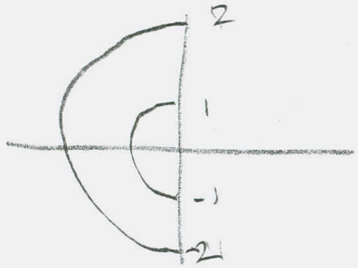
203 §16.4 #s 8, 11, 15, 18, 32, 36a

#s 7-14 Evaluate by changing to polar coordinates

8 $\iint_R (x+y)\,dA$, where R is to left of y-axis,

between $x^2+y^2=1$ & $x^2+y^2=4$



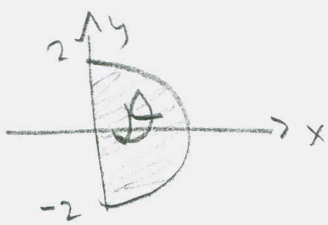
$$\int_{\theta=\frac{\pi}{2}}^{\theta=\frac{3\pi}{2}} \int_{r=1}^{r=2} (r\cos\theta + r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_{\theta=\frac{\pi}{2}}^{\theta=\frac{3\pi}{2}} (\cos\theta+\sin\theta)\,d\theta \int_{r=1}^{r=2} r^2\,dr =$$

$$\boxed{= -\frac{14}{3}}$$

$$= \Big[\sin\theta - \cos\theta\Big]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \Big[\tfrac{1}{3}r^3\Big]_{r=1}^{r=2} = \Big[(-1-0)-(1-0)\Big]\left(\tfrac{8}{3}-\tfrac{1}{3}\right)$$

11 $\iint_D e^{-x^2-y^2}\,dA$ D = region bdd by $x=\sqrt{4-y^2}$ & y-axis



$$\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{2} e^{-r^2}\, r\,dr\,d\theta$$

$-(x^2+y^2) = -r^2$ ✓

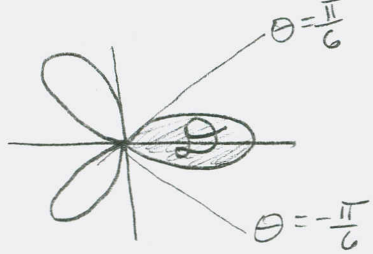
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^2 e^{-r^2} r\,dr = \Big[\theta\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Big[-\tfrac{1}{2}e^{-r^2}\Big]_0^2$$

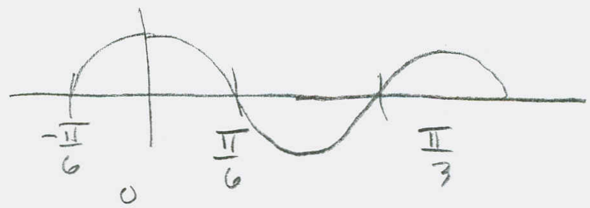
$$= (\pi)\left(-\tfrac{1}{2}\right)\left(e^{-4}-e^{0}\right) = \boxed{\frac{\pi}{2}\left(1-e^{-4}\right)}$$

203 § 16.4 #s 15, 18, 32, 36a

#s 15–18 Use iterated integral to find the area.

(15) One loop of the rose $r = \cos(3\theta)$





$r = 0$ to $r = \cos(3\theta)$

$\theta = -\frac{\pi}{6}$ to $\frac{\pi}{6}$

$$\iint_{\mathcal{D}} f(r,\theta)\, r\, dr\, d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_{r=0}^{r=\cos(3\theta)} 1\, r\, dr\, d\theta$$

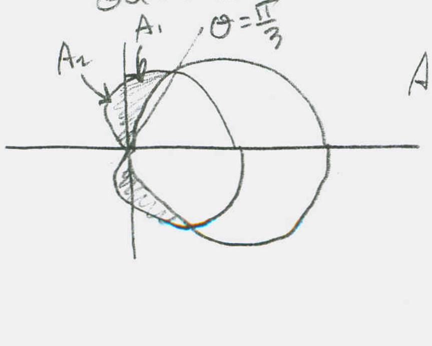
$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left[\tfrac{1}{2} r^2\right]_{r=0}^{r=\cos(3\theta)} d\theta = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2(3\theta)\, d\theta$$ EVEN FUNC

$$= \int_{0}^{\frac{\pi}{6}} \cos^2(3\theta)\, d\theta = \frac{1}{2}\int_{0}^{\frac{\pi}{6}} (1 + \cos(6\theta))\, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{6}\sin(6\theta)\right]_0^{\frac{\pi}{6}} = \frac{1}{2}\left[\left(\frac{\pi}{6} + 0\right) - (0 + 0)\right] = \boxed{\frac{\pi}{12}}$$

(18) Inside $r = 1 + \cos\theta$ (cardioid) &

outside $r = 3\cos\theta$ (circle)



$$A_1 = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_{3\cos\theta}^{1+\cos\theta} r\, dr\, d\theta$$

203 §16.4 #s 18, 32, 36a

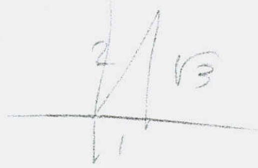
$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left[\frac{1}{2}r^2\right]_{3\cos\theta}^{1+\cos\theta} d\theta \quad = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left[(1+2\cos\theta+\cos^2\theta) - 9\cos^2\theta\right] d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1+2\cos\theta - 8\cos^2\theta)\, d\theta =$$

$8\cos^2\theta =$
$4(2\cos^2\theta) =$
$4(\cos(2\theta)+1)$
$= 4$

$$= \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1 + 2\cos\theta - 4\cos(2\theta) - 4)\, d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (-3 + 2\cos\theta - 4\cos(2\theta))\, d\theta$$



$$= \frac{1}{2}\left[-3\theta + 2\sin\theta - 2\sin(2\theta)\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= -\frac{3}{2}\left(\frac{\pi}{2} - \frac{\pi}{3}\right) + \left(\sin\frac{\pi}{2} - \sin\frac{\pi}{3}\right) - \left(\sin\pi - \sin\frac{2\pi}{3}\right)$$

$$= -\frac{\cancel{3}}{2}\left(\frac{\pi}{\cancel{6}\,2}\right) + \left(1 - \frac{\sqrt{3}}{2}\right) - \left(0 - \frac{\sqrt{3}}{2}\right) =$$

$$= -\frac{\pi}{4} + 1 - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \boxed{1 - \frac{\pi}{4}}$$

$$A_2 = \int_{\frac{\pi}{2}}^{\pi} \int_0^{1+\cos\theta} r\, dr\, d\theta = \int_{\frac{\pi}{2}}^{\pi} \left[\frac{1}{2}r^2\right]_{r=0}^{r=1+\cos\theta}$$

$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (1+2\cos\theta+\cos^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \left(1 + 2\cos\theta + \frac{1}{2}(\cos(2\theta)+1)\right) d\theta$$

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$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos(2\theta)\right) d\theta$$

$$= \frac{1}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin(2\theta)\right]_{\frac{\pi}{2}}^{\pi}$$

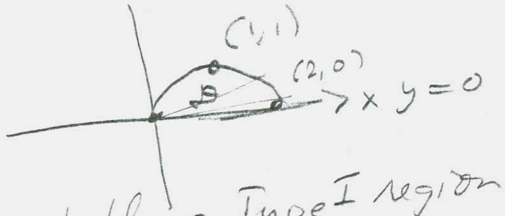
$$= \frac{1}{2}\left[\left(\frac{3}{2}\pi + (2)(0) + \frac{1}{4}(0)\right) - \left(\frac{3}{2}\left(\frac{\pi}{2}\right) + 2(1) + \frac{1}{4}(0)\right)\right]$$

$$= \frac{1}{2}\left[\frac{3}{2}\pi - \frac{3}{4}\pi + 2\right] = \frac{1}{2}\left[\frac{3}{4}\pi - 2\right] = \boxed{\frac{3}{8}\pi - 1}$$

Double $A_1 + A_2$ Gives $\boxed{\frac{\pi}{4} \text{ As FINAL}}$

(32) Convert to polar coords & evaluate

$$\int_{x=0}^{x=2}\int_{y=0}^{y=\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$



Evidently a Type I region

$y = \sqrt{2x-x^2}$

$y^2 = 2x - x^2$

$x^2 + y^2 = 2x$

$r^2 = 2r\cos\theta$

$r = 2\cos\theta$

$\sqrt{x^2+y^2} = \sqrt{r^2} = |r|$

Take r to be positive, which it is for $0 \le \theta \le \frac{\pi}{2}$

$$\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=2\cos\theta} r\; r\, dr\, d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=2\cos\theta} r^2\, dr\, d\theta$$

203 §16.4 #s 32, 36a

32 cnt'd

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left[\frac{1}{3}r^3\right]_0^{2\cos\theta} d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{1}{3}\left[(2\cos\theta)^3 - 0^3\right] d\theta$$

$$= \frac{8}{3}\int_{\theta=0}^{\theta=\frac{\pi}{2}} \cos^3\theta \, d\theta = \frac{8}{3}\int_{\theta=0}^{\theta=\frac{\pi}{2}} (1-\sin^2\theta)\cos\theta \, d\theta$$

$$= \frac{8}{3}\int_{\theta=0}^{\theta=\frac{\pi}{2}} \left(\cos\theta - \sin^2\theta\cos\theta\right) d\theta$$

$$= \frac{8}{3}\left[\sin\theta - \frac{1}{3}\sin^3\theta\right]_0^{\frac{\pi}{2}} = \frac{8}{3}\left[\left(\sin\frac{\pi}{2} - \frac{1}{3}\sin^3\frac{\pi}{2}\right) - (0-0)\right]$$

$$= \frac{8}{3}\left[1 - \frac{1}{3}\right] = \frac{8}{3}\left[\frac{2}{3}\right] = \boxed{\frac{16}{9}}$$

36a I = Improper integral over $\mathbb{R}^2$

$$= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dy\, dx$$

$$= \lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)} dA.$$ Show that this is π.

(D_a = disk with radius a & center = origin)

$$= \lim_{a\to\infty}\int_0^{2\pi}\int_0^a e^{-r^2} r\,dr\,d\theta = \lim_{a\to\infty}\int_0^{2\pi} d\theta \int_0^a e^{-r^2} r\,dr$$

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$$= [2\pi - 0] \lim_{a \to \infty} -\tfrac{1}{2} \left[e^{-r^2} \right]_0^a$$

$$= -\frac{2\pi}{2} \lim_{a \to \infty} \left(e^{-a^2} - 1 \right) = \pi$$

(b) $$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \lim_{a \to \infty} \iint_{S_a} e^{-(x^2+y^2)}\, dA$$

where S_a = square with vertices $(\pm a, \pm a)$.
use this to show that

$$\int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \pi \quad :$$

$$\iint_{S_a} e^{-(x^2+y^2)}\, dx\, dy = \pi = \int_{-a}^{a} \int_{-a}^{a} e^{-x^2} e^{-y^2} dx\, dy$$

$$= \int_{-a}^{a} e^{-x^2} dx \int_{-a}^{a} e^{-y^2} dy \xrightarrow{a \to \infty} \int_{-\infty} e^{-x^2} dx \int_{-\infty} e^{-y^2} dy$$

(We can play fast & loose with this because each of these integrals is bdd, by Calc II)

(c) By symmetry, we have $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

(d) $\therefore$ $\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt = \sqrt{2\pi}$.

App's in prob & stats, complex variables, Cauchy integral, Integration by residues