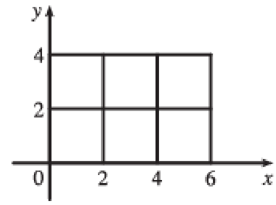
1. (a) The subrectangles are shown in the figure.



The surface is the graph of $f(x, y) = xy$ and $\Delta A = 4$, so we estimate

$$V \approx \sum_{i=1}^{3} \sum_{j=1}^{2} f(x_i, y_j)\, \Delta A$$

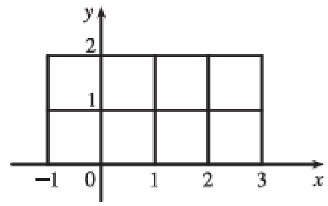
$$= f(2,2)\,\Delta A + f(2,4)\,\Delta A + f(4,2)\,\Delta A + f(4,4)\,\Delta A + f(6,2)\,\Delta A + f(6,4)\,\Delta A$$

$$= 4(4) + 8(4) + 8(4) + 16(4) + 12(4) + 24(4) = 288$$

(b) $V \approx \sum_{i=1}^{3} \sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\, \Delta A = f(1,1)\,\Delta A + f(1,3)\,\Delta A + f(3,1)\,\Delta A + f(3,3)\,\Delta A + f(5,1)\,\Delta A + f(5,3)\,\Delta A$

$$= 1(4) + 3(4) + 3(4) + 9(4) + 5(4) + 15(4) = 144$$

2. The subrectangles are shown in the figure.



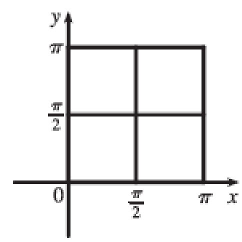
Since $\Delta A = 1$, we estimate

$$\iint_R (y^2 - 2x^2)\, dA \approx \sum_{i=1}^{4} \sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

$$= f(-1,1)\,\Delta A + f(-1,2)\,\Delta A + f(0,1)\,\Delta A + f(0,2)\,\Delta A$$
$$+ f(1,1)\,\Delta A + f(1,2)\,\Delta A + f(2,1)\,\Delta A + f(2,2)\,\Delta A$$

$$= -1(1) + 2(1) + 1(1) + 4(1) - 1(1) + 2(1) - 7(1) - 4(1) = -4$$

3. (a) The subrectangles are shown in the figure. Since $\Delta A = \pi^2/4$, we estimate

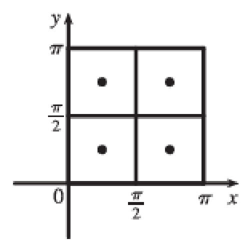


$$\iint_R \sin(x + y)\, dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

$$= f(0,0)\,\Delta A + f\left(0, \tfrac{\pi}{2}\right) \Delta A + f\left(\tfrac{\pi}{2}, 0\right) \Delta A + f\left(\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \Delta A$$

$$= 0\left(\tfrac{\pi^2}{4}\right) + 1\left(\tfrac{\pi^2}{4}\right) + 1\left(\tfrac{\pi^2}{4}\right) + 0\left(\tfrac{\pi^2}{4}\right) = \tfrac{\pi^2}{2} \approx 4.935$$

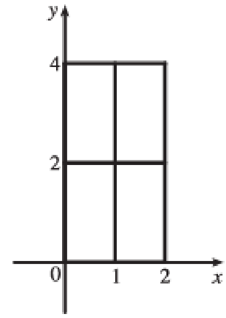
(b) $\iint_R \sin(x + y)\, dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\, \Delta A$



$$= f\left(\tfrac{\pi}{4}, \tfrac{\pi}{4}\right) \Delta A + f\left(\tfrac{\pi}{4}, \tfrac{3\pi}{4}\right) \Delta A + f\left(\tfrac{3\pi}{4}, \tfrac{\pi}{4}\right) \Delta A + f\left(\tfrac{3\pi}{4}, \tfrac{3\pi}{4}\right) \Delta A$$

$$= 1\left(\tfrac{\pi^2}{4}\right) + 0\left(\tfrac{\pi^2}{4}\right) + 0\left(\tfrac{\pi^2}{4}\right) + (-1)\left(\tfrac{\pi^2}{4}\right) = 0$$

4. (a) The subrectangles are shown in the figure.



The surface is the graph of $f(x, y) = x + 2y^2$ and $\Delta A = 2$, so we estimate

$$V = \iint_R (x + 2y^2)\, dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$$

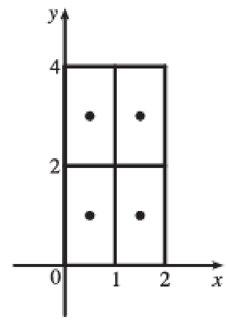
$$= f(1,0)\,\Delta A + f(1,2)\,\Delta A + f(2,0)\,\Delta A + f(2,2)\,\Delta A$$

$$= 1(2) + 9(2) + 2(2) + 10(2) = 44$$

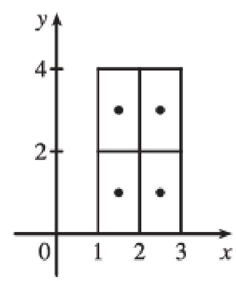
(b) $$V = \iint_R (x + 2y^2)\, dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\,\Delta A$$

$$= f\left(\tfrac{1}{2}, 1\right)\Delta A + f\left(\tfrac{1}{2}, 3\right)\Delta A + f\left(\tfrac{3}{2}, 1\right)\Delta A + f\left(\tfrac{3}{2}, 3\right)\Delta A$$

$$= \tfrac{5}{2}(2) + \tfrac{37}{2}(2) + \tfrac{7}{2}(2) + \tfrac{39}{2}(2) = 88$$



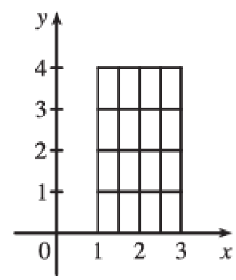
5. (a) Each subrectangle and its midpoint are shown in the figure. The area of each subrectangle is $\Delta A = 2$, so we evaluate f at each midpoint and estimate



$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right) \Delta A$$

$$= f(1.5, 1)\,\Delta A + f(1.5, 3)\,\Delta A$$
$$+\ f(2.5, 1)\,\Delta A + f(2.5, 3)\,\Delta A$$
$$= 1(2) + (-8)(2) + 5(2) + (-1)(2) = -6$$

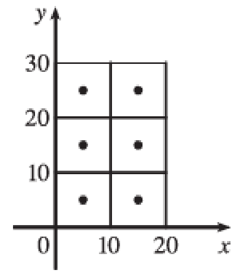
(b) The subrectangles are shown in the figure. In each subrectangle, the sample point farthest from the origin is the upper right corner, and the area of each subrectangle is $\Delta A = \frac{1}{2}$. Thus we estimate



$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{4} \sum_{j=1}^{4} f(x_i, y_j)\,\Delta A$$

$$= f(1.5, 1)\,\Delta A + f(1.5, 2)\,\Delta A + f(1.5, 3)\,\Delta A + f(1.5, 4)\,\Delta A$$
$$+ f(2, 1)\,\Delta A + f(2, 2)\,\Delta A + f(2, 3)\,\Delta A + f(2, 4)\,\Delta A$$
$$+ f(2.5, 1)\,\Delta A + f(2.5, 2)\,\Delta A + f(2.5, 3)\,\Delta A + f(2.5, 4)\,\Delta A$$
$$+ f(3, 1)\,\Delta A + f(3, 2)\,\Delta A + f(3, 3)\,\Delta A + f(3, 4)\,\Delta A$$
$$= 1\left(\tfrac{1}{2}\right) + (-4)\left(\tfrac{1}{2}\right) + (-8)\left(\tfrac{1}{2}\right) + (-6)\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right) + 0\left(\tfrac{1}{2}\right) + (-5)\left(\tfrac{1}{2}\right) + (-8)\left(\tfrac{1}{2}\right)$$
$$+ 5\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right) + (-1)\left(\tfrac{1}{2}\right) + (-4)\left(\tfrac{1}{2}\right) + 8\left(\tfrac{1}{2}\right) + 6\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right) + 0\left(\tfrac{1}{2}\right)$$
$$= -3.5$$

6. To approximate the volume, let R be the planar region corresponding to the surface of the water in the pool, and place R on coordinate axes so that x and y correspond to the dimensions given. Then we define $f(x, y)$ to be the depth of the water at (x, y), so the volume of water in the pool is the volume of the solid that lies above the rectangle $R = [0, 20] \times [0, 30]$ and below the graph of $f(x, y)$. We can estimate this volume using the Midpoint Rule with $m = 2$ and $n = 3$, so $\Delta A = 100$. Each subrectangle with its midpoint is shown in the figure. Then



$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{3} f(\overline{x}_i, \overline{y}_j)\, \Delta A = \Delta A[f(5,5) + f(5,15) + f(5,25) + f(15,5) + f(15,15) + f(15,25)]$$

$$= 100(3 + 7 + 10 + 3 + 5 + 8) = 3600$$

Thus, we estimate that the pool contains 3600 cubic feet of water.

Alternatively, we can approximate the volume with a Riemann sum where $m = 4$, $n = 6$ and the sample points are taken to be, for example, the upper right corner of each subrectangle. Then $\Delta A = 25$ and

$$V \approx \sum_{i=1}^{4} \sum_{j=1}^{6} f(x_i, y_j)\, \Delta A$$

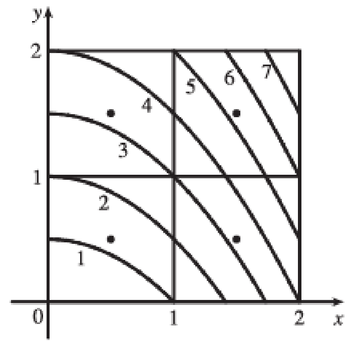
$$= 25[3 + 4 + 7 + 8 + 10 + 8 + 4 + 6 + 8 + 10 + 12 + 10 + 3 + 4 + 5 + 6 + 8 + 7 + 2 + 2 + 2 + 3 + 4 + 4]$$

$$= 25(140) = 3500$$

So we estimate that the pool contains 3500 ft^3 of water.

7. The values of $f(x, y) = \sqrt{52 - x^2 - y^2}$ get smaller as we move farther from the origin, so on any of the subrectangles in the problem, the function will have its largest value at the lower left corner of the subrectangle and its smallest value at the upper right corner, and any other value will lie between these two. So using these subrectangles we have $U < V < L$. (Note that this is true no matter how R is divided into subrectangles.)

8. Divide R into 4 equal rectangles (squares) and identify the midpoint of each subrectangle as shown in the figure.



The area of each subrectangle is $\Delta A = 1$, so using the contour map to estimate the function values at each midpoint, we have

$$\iint_R f(x,y)\; dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right)\Delta A = f\left(\tfrac{1}{2},\tfrac{1}{2}\right)\Delta A + f\left(\tfrac{1}{2},\tfrac{3}{2}\right)\Delta A + f\left(\tfrac{3}{2},\tfrac{1}{2}\right)\Delta A + f\left(\tfrac{3}{2},\tfrac{3}{2}\right)\Delta A$$

$$\approx (1.3)(1) + (3.3)(1) + (3.2)(1) + (5.2)(1) = 13.0$$

You could improve the estimate by increasing m and n to use a larger number of smaller subrectangles.

9. (a) With $m = n = 2$, we have $\Delta A = 4$. Using the contour map to estimate the value of f at the center of each subrectangle, we have

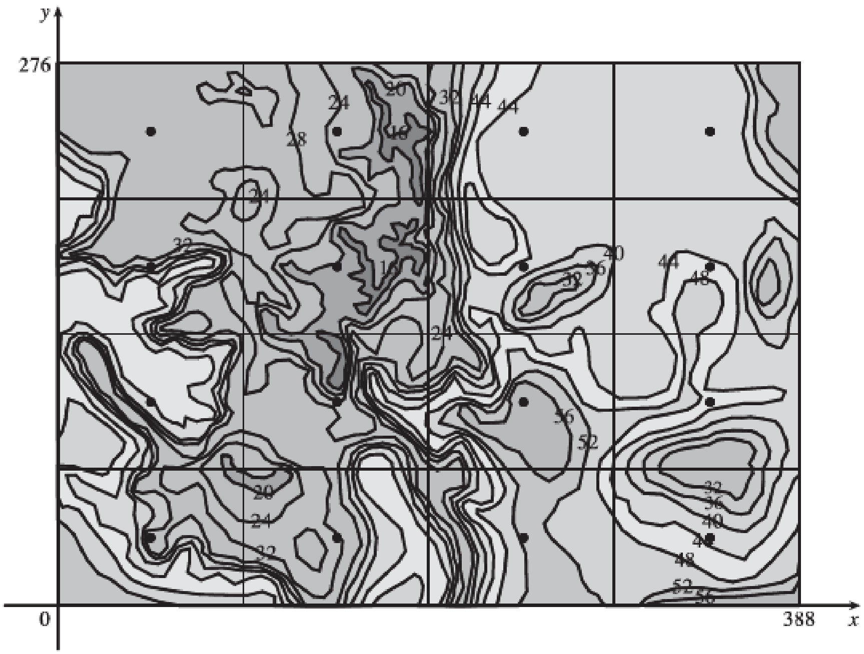
$$\iint_R f(x,y)\, dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right)\Delta A = \Delta A[f(1,1) + f(1,3) + f(3,1) + f(3,3)] \approx 4(27 + 4 + 14 + 17) = 248$$

(b) $f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x,y)\, dA \approx \frac{1}{16}(248) = 15.5$

10. As in Example 4, we place the origin at the southwest corner of the state. Then $R = [0, 388] \times [0, 276]$ (in miles) is the rectangle corresponding to Colorado and we define $f(x, y)$ to be the temperature at the location (x, y). The average temperature is given by

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\, dA = \frac{1}{388 \cdot 276} \iint_R f(x, y)\, dA$$

To use the Midpoint Rule with $m = n = 4$, we divide R into 16 regions of equal size, as shown in the figure, with the center of each subrectangle indicated.



The area of each subrectangle is $\Delta A = \frac{388}{4} \cdot \frac{276}{4} = 6693$, so using the contour map to estimate the function values at each midpoint, we have

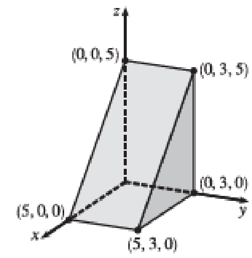
$$\begin{aligned} \iint_R f(x, y)\, dA \quad & \sum_{i=1}^{4} \sum_{j=1}^{4} f(\overline{x}_i, \overline{y}_j)\, \Delta A \\ & \approx \Delta A\, [31 + 28 + 52 + 43 + 43 + 25 + 57 + 46 + 36 + 20 + 42 + 45 + 30 + 23 + 43 + 41] \\ & = 6693(605) \end{aligned}$$

Therefore, $f_{\text{ave}} \approx \dfrac{6693 \cdot 605}{388 \cdot 276} \approx 37.8$, so the average temperature in Colorado at 4:00 PM on February 26, 2007, was approximately 37.8°F.

11. $z = 3 > 0$, so we can interpret the integral as the volume of the solid S that lies below the plane $z = 3$ and above the rectangle $[-2, 2] \times [1, 6]$. S is a rectangular solid, thus $\iint_R 3\, dA = 4 \cdot 5 \cdot 3 = 60$.

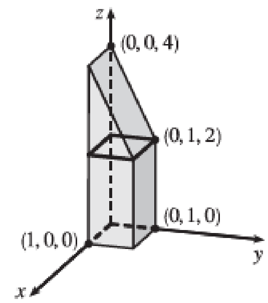
12. $z = 5 - x \geq 0$ for $0 \leq x \leq 5$, so we can interpret the integral as the volume of the solid S that lies below the plane $z = 5 - x$ and above the rectangle $[0, 5] \times [0, 3]$. S is a triangular cylinder whose volume is $3(\text{area of triangle}) = 3\left(\frac{1}{2} \cdot 5 \cdot 5\right) = 37.5$. Thus

$$\iint_R (5 - x)\, dA = 37.5$$

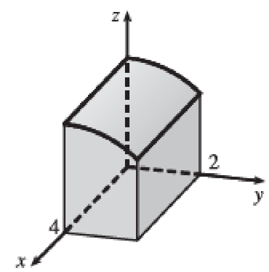


13. $z = f(x, y) = 4 - 2y \geq 0$ for $0 \leq y \leq 1$. Thus the integral represents the volume of that part of the rectangular solid $[0, 1] \times [0, 1] \times [0, 4]$ which lies below the plane $z = 4 - 2y$. So

$$\iint_R (4 - 2y)\, dA = (1)(1)(2) + \tfrac{1}{2}(1)(1)(2) = 3$$



14. Here $z = \sqrt{9 - y^2}$, so $z^2 + y^2 = 9$, $z \geq 0$. Thus the integral represents the volume of the top half of the part of the circular cylinder $z^2 + y^2 = 9$ that lies above the rectangle $[0, 4] \times [0, 2]$.



15. To calculate the estimates using a programmable calculator, we can use an algorithm similar to that of Exercise 5.1.7 [ET 5.1.7]. In Maple, we can define the function $f(x, y) = \sqrt{1 + xe^{-y}}$ (calling it `f`), load the `student` package, and then use the command

```
middlesum(middlesum(f,x=0..1,m),
                          y=0..1,m);
```

to get the estimate with $n = m^2$ squares of equal size. Mathematica has no special Riemann sum command, but we can define `f` and then use nested `Sum` commands to calculate the estimates.

n	estimate
1	1.141606
4	1.143191
16	1.143535
64	1.143617
256	1.143637
1024	1.143642

16.

n	estimate
1	0.934591
4	0.881991
16	0.865750

n	estimate
64	0.860490
256	0.858745
1024	0.858157

17. If we divide R into mn subrectangles, $\iint_R k\,dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$ for any choice of sample points $\left(x_{ij}^*, y_{ij}^*\right)$.

But $f\left(x_{ij}^*, y_{ij}^*\right) = k$ always and $\sum_{i=1}^{m} \sum_{j=1}^{n} \Delta A =$ area of $R = (b-a)(d-c)$. Thus, no matter how we choose the sample

points, $\sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A = k \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta A = k(b-a)(d-c)$ and so

$\iint_R k\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A = \lim_{m,n\to\infty} k \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta A = \lim_{m,n\to\infty} k(b-a)(d-c) = k(b-a)(d-c)$.

18. Because $\sin \pi x$ is an increasing function for $0 \le x \le \frac{1}{4}$, we have $\sin 0 \le \sin \pi x \le \sin \frac{\pi}{4} \quad \Rightarrow \quad 0 \le \sin \pi x \le \frac{\sqrt{2}}{2}$.

Similarly, $\cos \pi y$ is a decreasing function for $\frac{1}{4} \le y \le \frac{1}{2}$, so $0 = \cos \frac{\pi}{2} \le \cos \pi y \le \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$. Thus on R,

$0 \le \sin \pi x \, \cos \pi y \le \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{1}{2}$. Property (9) gives $\iint_R 0\,dA \le \iint_R \sin \pi x \cos \pi y\,dA \le \iint_R \frac{1}{2}\,dA$, so by Exercise 17 we

have $0 \le \iint_R \sin \pi x \, \cos \pi y\,dA \le \frac{1}{2}\left(\frac{1}{4} - 0\right)\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{32}$.