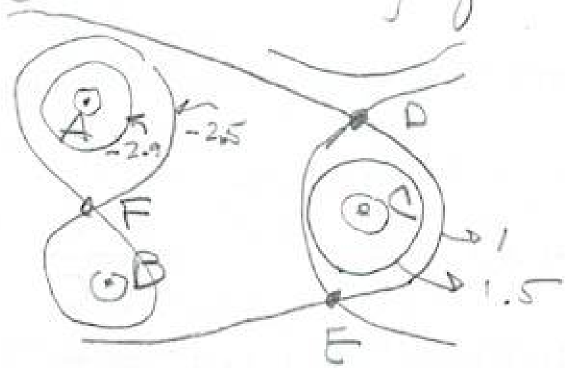
2.03 §14.7 #s 3, 6, 11, 19, 21, 26, 30, 33, 39, 40

(4) Use level curves to predict critical points & if it's max/min/saddle

I think I messed up the #s.

Oh well. Here's my quick take on #4



A, B MIN

C MAX

D, E, F SADDLE

A(-1, 1) D(1, 1)

B(-1, -1) E(1, -1)

C(1, 0) F(-1, 0)

#s 5–18 Find local extreme values & saddle points. Graph.

(6) $f(x,y) = x^3y + 12x^2 - 8y$

$f_x = 3x^2y + 24x \overset{\text{set}}{=} 0 \Rightarrow 3x(xy + 24) = 0 \Rightarrow \underline{x = 0},\ x = -\frac{24}{y}$

$y = -\frac{24}{x}$

$f_y = x^3 - 8 = 0 \Rightarrow x = 2$

$y = \frac{-24}{2} = -12 \rightarrow (2, -12)$

$f_{xx} = 6xy - 24,\ f_{yy} = 0,\ f_{xy} = 3x^2$

$D < 0$ everywhere, so

Not max/min

$(2, -12, f(2,-12))$ is saddle

203 §14.7 #s 6, 11, 19, 21, 26, 30, 33, 39, 40

(6) $f(x,y) = x^3y + 12x^2 - 8y$ Find local extrema or saddle pts.

$f_x = 3x^2y + 24x \overset{SET}{=} 0 \Rightarrow 3x(xy+8) \overset{SET}{=} 0$

$\Rightarrow x = 0$ OR $xy = -8 \Rightarrow x = -\frac{8}{y}$

$f_y = x^3 - 8 \overset{SET}{=} 0 \Rightarrow x = 2$

So $\left(x = 0 \text{ OR } x = -\frac{8}{y}\right)$ AND $x = 2$

$\Rightarrow \cancel{x = 0 \text{ and } x = 2}$ OR $x = -\frac{8}{y}$ & $x = 2$

$\Rightarrow 2 = -\frac{8}{y} \Rightarrow y = -4$

$(2,-4)$ is critical

$$\left.\begin{array}{l} f_{xx} = 6xy + 24 \\ f_{yy} = 0 \\ f_{xy} = 3x^2 \end{array}\right\} D = 0 - (3x^2)^2 = -9x^4 < 0 \ \forall x \neq 0$$

So $f(2,-4) = 2^3(-4) + 12(2)^2 - 8(-4)$

$= -32 + 48 - 8 = 8$

$\boxed{(2,-4,8) \text{ is saddle pt.}}$

203 S' 14.7 #s ~~3~~, 11, 19, 21, 26, 30, 33, 39, 48

(11) $f(x,y) = x^3 - 12xy + 8y^3$

$f_x = 3x^2 - 12y \overset{SET}{=} 0 \Rightarrow 12y = 3x^2 \Rightarrow y = \frac{1}{4}x^2$

$f_y = -12x + 24y^2 \overset{SET}{=} 0 \Rightarrow x = 2y^2$

Combine: $y = \frac{1}{4}(\cancel{+}2y^2)^2 = y^4 \Rightarrow$

$y^4 - y = y(y^3 - 1) = y(y-1)(y^2+y+1) \overset{SET}{=} 0$

$\Rightarrow y = 0, 1 \Rightarrow$ $D = (6x)(48y) - 144$

$x = -2(0)^2 = 0 \rightsquigarrow (0,0)$

$x = \cancel{+}2(1)^2 = +2 \rightsquigarrow (+2, 1)$

$f_{xx} = 6x, \ f_{yy} = 48y, \ f_{xy} = -12 \Rightarrow$

$D(0,0) = 0 - (-12) = 12 > 0$ & $f_{xx} = 0 \Rightarrow ???$

$D(-2,1) = 6(+2)(48(1)) - (-12)^2 = 12(48) - 12(12)$

$= 36(12) > 0 \quad f_{yx}(2,1) = 6(2) = 12 > 0$

$\Rightarrow$ MIN

$f(0,0) = 0$

$f(2,1) = (+2)^3 - 12(+2)(1) + 8(1)^3$

$= +8 + 24 + 8 = 40$

$\boxed{\begin{array}{l} \text{SADDLE } \cancel{\text{M}} \ 0 \text{ @ } (0,0) \\ \text{MIN OF } 40 \text{ @ } (+2,1) \end{array}}$

$(0,0,0)$ = Saddle

$(2,1,40)$ = MIN

203 § 14.7 #s 19, 21, 26, 30, 33, 39, 40

(19) Show that $f(x,y) = x^2+4y^2-4xy+2$ has an ∞ # of critical pts & local (and absolute!) min @ each.

$f_x = 2x-4y \overset{SET}{=} 0 \Rightarrow x = 2y$

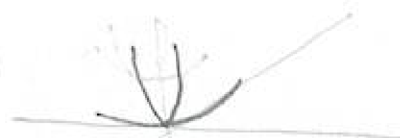
$f_y = 8y-4x \overset{SET}{=} 0 \Rightarrow x = 2y$

So everything on the plane $y = \frac{1}{2}x$ is a critical point.

$f_{xx} = 2,\ f_{yy} = 8,\ f_{xy} = -4 \Rightarrow$

$D(x,y) = (2)(8) - (-4)^2 = 0$ tells us nothing.

f_{xx} & $f_{yy} > 0$, so that's a good sign.



Along $y = \frac{1}{2}x$, we have

$f(x,y) = f(x, \frac{1}{2}x) = x^2 + 4(\frac{1}{2}x)^2 - 4(x)(\frac{1}{2}x) + 2$

$= x^2 + x^2 - 2x^2 + 2 = 2$ on $y = \frac{1}{2}x$.

Also, $f_{xx}, f_{yy} > 0 \Rightarrow f$ lies above its tangent plane on $y = \frac{1}{2}x$ & the only way for it to get back BELOW it would be for it to have a local max somewhere which it can't, since $y = \frac{1}{2}x$ contains all its critical points.

$\therefore$ along $y = \frac{1}{2}x$, they're all local mins.

203 § 14.7 #s 21, 26, 30, 33, 39, 40

(21) Use a graph or level curves to estimate local max / min / saddle pts. Then use calc. to find them.

$f(x,y) = x^2 + y^2 + \frac{1}{x^2y^2} = x^2 + y^2 + x^{-2}y^{-2}$

$f_x = 2x - 2x^{-3}y^{-2} = 0 \Rightarrow x - \frac{1}{x^3y^2} =$

$\frac{x^4y^2 - 1}{x^3y^2} = 0 \Rightarrow x^4y^2 - 1 = 0$

$f_y = 2y - 2x^{-2}y^{-3} = 0 \rightarrow \frac{y}{1} \cdot \frac{x^2y^3}{x^2y^3} - \frac{1}{x^2y^3}$

$= \frac{x^2y^4 - 1}{x^2y^3} = 0 \Rightarrow x^2y^4 - 1 = 0$

$\Rightarrow x^4y^2 = 1 \qquad y^4x^2 = 1$

$y^2 = \frac{1}{x^4} \qquad x = \pm\frac{1}{y^2}$

$y = \pm\frac{1}{x^2} = \pm\frac{1}{(\pm\frac{1}{y^2})^2} = \pm\frac{1}{y^4}$

$\frac{y^5 \mp 1}{y^4} = 0 \rightarrow (y-1)(y^4 + y^3 + y^2 + y + 1) = 0$

$y = 1 \rightarrow x = \pm 1$

$\boxed{(-1, 1), (1, 1)}$

$$\begin{array}{r|rrrrrr} 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ & & 1 & 1 & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 1 & 1 & 0 \end{array}$$

$(y+1)(y^4 - y^3 + y^2 - y + 1)$

$$\begin{array}{r|rrrrrr} -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ & & -1 & 1 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & -1 & 1 & 0 \end{array}$$

$y = -1 \rightarrow x = \pm 1$

$\boxed{(1, -1), (-1, 1)}$

203 §14.7 #s 21, 26, 30, 33, 39, 40

21 cont'd

$$f_{xx} = 2 + 6x^{-4}y^{-2}$$

$$f_{yy} = 2 + 6x^{-2}y^{-4}$$

$$f_{xy} = 4x^{-3}y^{-3}$$

$$D(x,y) = (2+6x^{-4}y^{-2})(2+6x^{-2}y^{-4}) - 16x^{-6}y^{-6}$$

@ $(\pm 1, \pm 1)$, $(\pm 1, \mp 1)$:

$$D(x,y) = (8)(8) - 16 > 0$$

$$f_{xx}(\pm 1, \pm 1) = 8 > 0 \Longrightarrow \text{mins}$$

$$f(\pm 1, \pm 1) = 1^2 + 1^2 + \frac{1}{1^2 1^2} = 3$$

$f(\pm 1, \mp 1) = 3$ as well

So, min val. of $f(x,y) = 3$ @ $(\pm 1, \pm 1)$, $(\pm 1, \mp 1)$

26 Use tech to find critical pts. to 3 decimal places. Then classify them & find highs/lows of graph

$$f(x,y) = 5 - 10xy - 4x^2 + 3y - y^4$$

$$f_x = -10y - 8x \overset{\text{set}}{=} 0 \longrightarrow y = -\frac{4}{5}x$$

$$f_y = -10x + 3 - 4y^3 \overset{\text{set}}{=} 0 \longrightarrow -10x = 4y^3 - 3$$

$$\longrightarrow x = -\frac{2}{5}y^3 + \frac{3}{10} \longrightarrow y = -\frac{4}{5}\left(-\frac{2}{5}y^3 + \frac{3}{10}\right)$$

$$\longrightarrow 5y = \frac{8}{5}y^3 - \frac{6}{5} \longrightarrow 25y = 8y^3 - 6$$

203 §14.7 #s 26, 30, 33, 39, 40

#26 cont'd

$$8y^3 - 25y - 6 = 0$$

$\pm 6, \pm \frac{6}{2} = 3, \pm \frac{6}{4} = \frac{3}{2}, \pm \frac{6}{8} = \frac{3}{4}$

$\pm 3, \pm \frac{3}{2}, \pm \frac{3}{4}, \pm \frac{3}{8}$

$\pm 2, \pm \frac{2}{2} = 1, \pm \frac{2}{4} = \frac{1}{2}, \pm \frac{2}{8} = \frac{1}{4}$

$\pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm \frac{1}{8}$

No rational roots (checked)

$y \approx -1.632676, -.244688, 1.8773642$

use $x = -\frac{2}{5} y^3 + \frac{3}{10}$ to find x's

$x \approx 2.040844652, .3058600052, -2.3467053$

$(2.041, -1.633), (.306, -.245), (-2.347, 1.877)$

Ugh. Needs Maple.

203 §14.7 #s 30, 33, 39, 40

#s 29-36 Find Abs Max & Min of f on $\mathcal{D}$

30 $f(x,y) = 3 + xy - x - 2y$

$\mathcal{D}$ = Closed triangle w/ vertices

$(1,0), (5,0), (1,4)$

$(1,0), (5,0)$: $y = 0$ $\quad L_1$

$(1,0), (1,4)$: $x = 1$ $\quad L_2$

$(5,0), (1,4)$: $m = \frac{4}{-4} = -1$: $y = -1(x-5) = -x+5$ $\quad L_3$

L_1: $f(x,0) = 3 - x \qquad 1 \le x \le 5$

$$\boxed{\begin{aligned} f(1,0) &= 3-1 = 2 \\ f(5,0) &= 3-5 = -2 \end{aligned}}$$

L_2: $f(1,y) = 3 + y - 1 - 2y = 2 - y \qquad 0 \le y \le 4$

$\boxed{f(1,0) = 2}$

$f(1,4) = 3 + (1)(4) - 1 - 2(4) = \boxed{-2 = f(1,4)}$

L_3: $f(x,-x+5) = 3 + x(-x+5) - x - 2(-x+5)$

$= 3 - x^2 + 5x - x + 2x - 10 = -x^2 + 6x - 7 = g(x)$

$1 \le x \le 5$

$-2x + 6 = 0$

$2x = 6$

$x = 3 \qquad g(3) = -9 + 18 - 7 = \boxed{2 = f(3,2)}$

$g(1) = g(5) = -1 + 6 - 7 = \boxed{-2 = f(1,4)}$

203 §14.7 #s 30, 33, 39, 40

$\boxed{\#30 \text{ cnt'd}}$

$f_x = y - 1 \overset{\text{SET}}{=} 0 \Rightarrow y = 1$ $\Big\}$ $(1,2)$ critical

$f_y = x - 2 \Rightarrow x = 2$

$f_{xx} = 0, \; f_{yy} = 0, \; f_{xy} = 1$

$D(\text{ANY}) = -1 < 0 \Rightarrow (1,2)$ is saddle pt

$f(1,2) = 3 + (1)(2) - 1 - 2(2) = 0$

$f(1,2) = 0$ saddle pt

otherwise, absolute max on $\mathcal{D}$ of $f(x,y) = 2$ @ $(1,0), (3,2)$ & Absolute min @ $(5,0), (1,4)$ of $f(x,y) = -2$

(33) $f(x,y) = x^4 + y^4 - 4xy + 2$ on $\mathcal{D}$

$= \text{rectangle} = \{(x,y) : 0 \le x \le 3, \; 0 \le y \le 2\}$

$\mathcal{L}_1 : x = 0, \; 0 \le y \le 2, \quad \mathcal{L}_2 : x = 3, \; 0 \le y \le 2$

$\mathcal{L}_3 : y = 0, \; 0 \le x \le 3, \quad \mathcal{L}_4 : y = 2, \; 0 \le x \le 3$

$\boxed{f(0,0) = 2, \; f(0,2) = 18}$

$\mathcal{L}_1 : f(0,y) = y^4 + 2, \; 0 \le y \le 2$

min of 2, max of 18

$0 \le y \le 3$

$\mathcal{L}_2 : f(3,y) = 3^4 + y^4 - 4(3)y + 2 = y^4 - 12y + 83$ $\quad 0 \le y \le 2$

$f_y = 4y^3 - 12 = 4(y^3 - 3) \overset{\text{SET}}{=} 0 \Rightarrow y = \sqrt[3]{3}$

$\boxed{f(3,0) = 83}$ $\quad f(3,2) = 81 + 16 - 24 + 2$ $\boxed{= \cancel{75} = f(3,2)}$

NEED $f(3, \sqrt[3]{3})$

203 §14.7 #s ~~30~~, 33, 39, 40

$\boxed{\#33 \text{ cnt'd}}$

ℓ_3: $f(x,0)$ $\quad 0 \le x \le 3$

$= x^4 + 2$ $\quad \boxed{f(0,0) = 2}$ $\quad \boxed{f(3,0) = 3^4 + 2 = 83}$

ℓ_4: $f(x,2) = x^4 + 2^4 - 4(2)x + 2 \quad 0 \le x \le 3$

$= x^4 - 8x + 18 = z$

$z' = 4x^3 - 8 = 4(x^3 - 2) \overset{\text{set}}{=} 0 \implies x = \sqrt[3]{2}$

$f(0,2) = 2^4 + 2 = 16 + 2 = \boxed{18 = f(0,2)}$

$f(3,2) = 3^4 + 2^4 - 4(3)(2) + 2$

$= 81 + 16 - 24 + 2 = \boxed{75 = f(3,2)}$

Spares:

$f(3, \sqrt[3]{3}) = 3^4 + \sqrt[3]{3}^4 - 4(3)(\sqrt[3]{3}) + 2$

$= 81 + 3\sqrt[3]{3} - 12\sqrt[3]{3} + 2 = 83 - 9\sqrt[3]{3}$

≈ 70.0197387

$f(\sqrt[3]{2}, 2) = \sqrt[3]{2}^4 + 2^4 - 4(\sqrt[3]{2})(2) + 2 = 18 - 6\sqrt[3]{2}$

$= 2\sqrt[3]{2} + 16 - 8\sqrt[3]{2} + 2$

≈ 10.440737

$$\boxed{\begin{array}{l} f(3, \sqrt[3]{3}) \approx 70.0197 \\ f(\sqrt[3]{2}, 2) \approx 10.440737 \end{array}}$$

203 $\S 14.7$ #s 33, 39, 40

#33 cnt'd

$f(x,y) = x^4 + y^4 - 4xy + 2$

$f_x = 4x^3 - 4y \overset{SET}{=} 0 \Rightarrow y = x^3$

$f_y = 4y^3 - 4x \overset{SET}{=} 0 \Rightarrow x = y^3$

$\Big\} \; x = y$

$y = x \rightarrow f(x,y) = f(x) = x^4 + x^4 - 4x^2 + 2$

$= 2x^4 - 4x^2 + 2 = 2(x^4 - 2x^2 + 1)$

$f_{xx} = 12x^2$

$f_{yy} = 12y^2$

$f_{xy} = -4$

$\Big\} \; D(x,y) = 144x^2y^2 > 0 \ \forall (x,y) \ni (x,y) \neq (0,0)$

$f_{xx} \geq 0 \ \forall (x,y). \ f_{xx} > 0 \ \forall (x,y) \neq (0,0)$

$\Rightarrow$ min.

$\boxed{f(0,0) = 2}$

$f(0,2) = 18$

$f(3,2) = 75$

$f(3, \sqrt[3]{2}) \approx 70.02$

$f(\sqrt[3]{2}, 2) \approx 10.44$

$\boxed{\begin{array}{l} f(3,0) = 83 \quad \text{MAX} \\ f(0,0) = 2 \quad \text{MIN} \end{array}}$

203 §14.7 #s 39, 40

(39) Find the shortest distance from $(2,1,-1) = A$

to $x + y - z = 1$ NOTE $B(1,0,0) \in P$

$\overrightarrow{AB} = \bar{u} = \langle -1, -1, 1 \rangle$

$d = |\text{comp}_{\bar{n}} \bar{u}| = \frac{|\bar{n} \cdot \bar{u}|}{\|\bar{n}\|} = \frac{\langle 1, 1, -1 \rangle \cdot \langle -1, -1, 1 \rangle}{\sqrt{3}}$

$= \frac{|-1-1-1|}{\sqrt{3}} = \frac{3}{\sqrt{3}}$ OR $\frac{3\sqrt{3}}{3} = \boxed{\sqrt{3} = d}$

FORMULA :

$\frac{|\langle 1,1,1 \rangle \cdot \langle 2,1,-1 \rangle + 1|}{\sqrt{3}} = \frac{|2+1-1+1|}{\sqrt{3}}$ ✓

Calculus :

$d = \sqrt{(x-2)^2 + (y-1)^2 + (z+1)^2}$

$z = x + y - 1$

$= \sqrt{(x-2)^2 + (y-1)^2 + (x+y-1+1)^2}$

$= \sqrt{(x-2)^2 + (y-1)^2 + (x+y)^2}$ →

$d^2 \equiv f(x,y) = (x-2)^2 + (y-1)^2 + (x+y)^2$

$f_x = 2(x-2) + 2(x+y) = 2x - 4 + 2x + 2y = 4x + 2y - 4$

$f_y = 2(y-1) + 2(x+y) = 2y - 2 + 2x + 2y = 2x + 4y - 2$

$\left[\begin{array}{cc|c} 2 & 4 & 2 \\ 4 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -3 & 0 \end{array}\right]$

$y = 0, x = 1$

$f(1,0) = 1^2 + 1^2 + 1^2 = 3$

$d = \sqrt{f} = \boxed{\sqrt{3}}$ ✓

203 §14.7 #40

(40) Find closest $P(x_0, y_0, z_0)$ to

$A(1,2,3)$ on $P: x - y + z = 4$

$$d = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$$

minimize $f(x,y) = (x-1)^2 + (y-2)^2 + (1-x+y)^2$

from $z = 4 - x + y \Rightarrow z - 3 = 1 - x + y$

$\Rightarrow f_x = 2(x-1) - 2(1-x+y) = 2x - 2 - 2 + 2x - 2y$

$= 4x - 2y - 4 \overset{SET}{=} 0 \Rightarrow \underline{2x - y = 2}$

$f_y = 2(y-2) + 2(1-x+y) = 2y - 4 + 2 - 2x + 2y$

$= -2x + 4y - 2 \overset{SET}{=} 0 \Rightarrow -2x + 4y = 2$

$\Rightarrow \underline{x - 2y = -1}$ so

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ 2 & -1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 3 & 4 \end{array}\right]$$

$\Rightarrow \boxed{y = \frac{4}{3}} \Rightarrow x - 2\left(\frac{4}{3}\right) = -1 \Rightarrow x = \frac{8-3}{3} = \boxed{\frac{5}{3} = x}$

$\boxed{x = \frac{5}{3}} \Rightarrow z = 4 - x + y = 4 - \frac{5}{3} + \frac{4}{3}$

$= \frac{12-5+4}{3} = \boxed{\frac{11}{3} = z}$

So $\boxed{\left(\frac{5}{3}, \frac{4}{3}, \frac{11}{3}\right)}$ is closest. There we

So

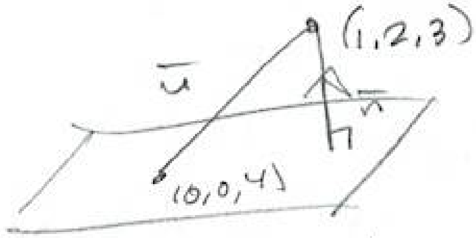
have $d = \sqrt{\left(\frac{5}{3}-1\right)^2 + \left(\frac{4}{3}-2\right)^2 + \left(\frac{11}{3}-3\right)^2} = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{4}{9}} = \sqrt{\frac{12}{9}} = \frac{2\sqrt{3}}{3}$

Check!

$\left(\frac{5}{3}, \frac{4}{3}, \frac{11}{3}\right)$

$(0,0,4) \in P$

$x - y + z = 4$



$\bar{u} = \langle 1, 2, -1 \rangle$

$$|\text{comp}_{\bar{n}} \bar{u}| = \frac{|\bar{n} \cdot \bar{u}|}{\|\bar{n}\|} = \frac{\langle 1,-1,1\rangle \cdot \langle 1,2,-1\rangle}{\sqrt{3}}$$

$$= \frac{|1 - 2 - 1|}{\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$