203 §13.3 #s 1, 4, 7, 8, 10, 13, 21, 26, 47, 48

#s 1-6 Find the length of the curve.

(1) $\langle t, 3\cos t, 3\sin t\rangle = \bar{r} \quad -5 \leq t \leq 5$

$\bar{r}' = \langle 1, -3\sin t, 3\cos t\rangle$

$$L = \int_{-5}^{5} \sqrt{1 + 9\sin^2 t + 9\cos^2 t}\, dt = \int_{-5}^{5} \sqrt{10}\, dt$$

$$= \sqrt{10}\left[t\right]_{-5}^{5} = \sqrt{10}(5) - \sqrt{10}(-5) = \boxed{10\sqrt{10}}$$

(4) $\langle \cos t, \sin t, \ln(\cos t)\rangle \quad t \in [0, \frac{\pi}{4}]$

$\bar{r}' = \langle -\sin t, \cos t, \frac{-\sin t}{\cos t}\rangle$

$$L = \int_{0}^{\frac{\pi}{4}} \sqrt{\sin^2 t + \cos^2 t + \tan^2 t}\, dt = \int_{0}^{\frac{\pi}{4}} \sqrt{1 + \tan^2 t}\, dt$$

$$= \int_{0}^{\frac{\pi}{4}} |\sec t|\, dt = \int_{0}^{\frac{\pi}{4}} \sec t\, dt$$

$$= \ln|\sec t + \tan t|\Big]_{0}^{\frac{\pi}{4}} = \ln\left(\sec\tfrac{\pi}{4} + \tan\tfrac{\pi}{4}\right) - \ln(1)$$

$$= \boxed{\ln(\sqrt{2} + 1)}$$

203 §13.3 #s 7, 8, 10, 13, 21, 26, 47, 48

#s 7–10 Find length, to 4 decimal places.

(7) $\bar{r}(t) = \langle t^2, t^3, t^4 \rangle$, $0 \le t \le 2$

$\bar{r}' = \langle 2t, 3t^2, 4t^3 \rangle$

$$\int_0^2 \sqrt{4t^2 + 9t^4 + 16t^6}\, dt \approx 18.683298$$

$$\boxed{\approx 18.6833}$$

(8) $\bar{r} = \langle t, e^{-t}, te^{-t} \rangle$, $1 \le t \le 3$

$r'(t) = \langle 1, -e^{-t}, e^{-t} - te^{-t} \rangle$

$$\int_1^3 \sqrt{1 + e^{-2t} + e^{-2t} - 2te^{-2t} + t^2e^{-2t}}\, dt$$

$$= \int_1^3 \sqrt{1 + 2e^{-2t} - 2te^{-2t} + t^2e^{-2t}}\, dt \approx 2.0453716$$

$$\boxed{\approx 2.0454}$$

(10) Graph $\langle \sin t, \sin 2t, \sin 3t \rangle$
Find total length of this curve to 4 places

$$\int_0^{2\pi} \sqrt{\cos^2 t + 4\cos^2 2t + 9\cos^2 3t}\, dt \approx 16.026401$$

$$\boxed{\approx 16.0264}$$

=

203 §13.3 #s 13, 21, 26, 47, 48

(13) (a) Find arc length from P in direction of increasing t & then re-parametrize wrt arc length starting from P.

(b) Find the point 4 units along the curve from P.

$\bar{r} = \langle 5-t, 4t-3, 3t \rangle \quad P(4,1,3) \rightarrow t=1$

$$s(t) = \int_1^t \sqrt{1^2+4^2+3^2}\,dt = \int_1^t \sqrt{1+16+9}\,dt$$

$$= \sqrt{26}\int_1^t dt = \sqrt{26}\,[t]_1^t = \sqrt{26}\,(t-1) = \sqrt{26}t - \sqrt{26}$$

$$\Rightarrow s = \sqrt{26}\,t - \sqrt{26} \Rightarrow \frac{s+\sqrt{26}}{\sqrt{26}} = t \rightarrow$$

$$\bar{r} = \left\langle 5 - \frac{s+\sqrt{26}}{\sqrt{26}},\ 4\left(\frac{s+\sqrt{26}}{\sqrt{26}}\right) - 3,\ 3\left(\frac{s+\sqrt{26}}{\sqrt{26}}\right) \right\rangle$$

$$s = 4 = \sqrt{26}\,t - \sqrt{26} \rightarrow$$

$$\frac{\sqrt{26}+4}{\sqrt{26}} = t \Rightarrow Q = \left(5 - \left(\frac{\sqrt{26}+4}{}\right)\right)$$

$$Q = \left(5 - \left(\frac{\sqrt{26}+4}{\sqrt{26}}\right),\ 4\left(\frac{\sqrt{26}+4}{\sqrt{26}}\right) - 3,\ 3\left(\frac{\sqrt{26}+4}{\sqrt{26}}\right)\right)$$

$$= \left(\frac{4\sqrt{26}-4}{\sqrt{26}},\ \frac{4\sqrt{26}+16-3\sqrt{26}}{\sqrt{26}},\ \frac{3\sqrt{26}+12}{\sqrt{26}}\right)$$

$$= \left(\frac{4\sqrt{26}-4}{\sqrt{26}},\ \frac{\sqrt{26}+16}{\sqrt{26}},\ \frac{3\sqrt{26}+12}{\sqrt{26}}\right)$$

203 §13.3 #s 21, 26, 47, 48

21 $\vec{r} = \langle 0, t^3, t^2 \rangle$ Find curvature.

$$\kappa = \left\| \frac{T'}{\vec{r}\,'} \right\|$$

$$\vec{r}\,' = \langle 0, 3t^2, 2t \rangle \quad \|\vec{r}\,'\| = \sqrt{9t^4 + 4t^2}$$

$$T = \frac{\langle 0, 3t^2, 2t \rangle}{\sqrt{9t^4+4t^2}} = \left(9t^4+4t^2\right)^{-\frac{1}{2}} \langle 0, 3t^2, 2t \rangle$$

$$\Rightarrow T' = -\tfrac{1}{2}(9t^4+4t^2)^{-\frac{3}{2}}(36t^3+8t)\langle 0, 3t^2, 2t \rangle$$

$$+ (9t^4+4t^2)^{-\frac{1}{2}} \langle 0, 6t, 2 \rangle$$

$$= \frac{-(36t^3+8t)}{2(9t^4+4t^2)^{3/2}} \langle 0, 3t^2, 2t \rangle + \frac{2(9t^4+4t^2)}{2(9t^4+4t^2)^{3/2}} \langle 0, 6t, 2 \rangle$$

$$= \frac{-(36t^3+8t)\langle 0, 3t^2, 2t \rangle + (18t^4+8t^2)\langle 0, 6t, 2 \rangle}{2(9t^4+4t^2)^{3/2}}$$

$$= \frac{\langle 0, -108t^5 - 24t^3, -72t^4 - 16t^2 \rangle + \langle 0, 108t^5 + 48t^3, 36t^4 + 16t^2 \rangle}{\text{Denom}}$$

$$= \frac{\langle 0, 24t^3, -36t^4 \rangle}{2(9t^4+4t^2)^{3/2}}$$

$$\frac{\|T'\|}{\|\vec{r}\,'\|} = \left\| \frac{\langle 0, 24t^3, -36t^4 \rangle}{2(9t^4+4t^2)^{3/2}} \right\| \Big/ \|\langle 0, 3t^2, 2t \rangle\|$$

203 §13.3 #s 21, 26, 47, 48

#21 cnt'd

$$\frac{\left(\frac{1}{2(9t^4+4t^2)^{3/2}}\right)\sqrt{24^2t^6+36^2t^8}}{\sqrt{9t^4+4t^2}}$$

$2\underline{|24}$, $2\underline{|12}$, $2\underline{|6}$, 3

$2\underline{|36}$, $2\underline{|18}$, $3\underline{|9}$, 3

$$=\left(\frac{1}{2(9t^4+4t^2)}\right)12t^3\sqrt{2^2+3^2t^2}$$

$$=\frac{12t^3\sqrt{9t^2+4}}{2t^2(9t^2+4)}=\boxed{6t\left(\frac{1}{\sqrt{9t^2+4}}\right)}$$

wow! Got it!

$\frac{\|\bar{r}'\times r''\|}{\|\bar{r}'\|^3}$: $\bar{r}''=\langle 0,9t,2\rangle$ so,

$\bar{r}'=\langle 0,3t^2,2t\rangle,0,3t^2$

$\bar{r}''=\langle 0,6t,2\rangle,0,9t$ $\longrightarrow$

$\langle 6-6t^2,0,0\rangle$

$$\frac{\|\bar{r}'\times\bar{r}''\|}{\|\bar{r}'\|^3}=\frac{\sqrt{36t^4}}{(\sqrt{9t^4+4t^2})^3}=\boxed{\frac{6t^2}{\sqrt{9t^4+4t^2}}}\ !$$

203 §13.3 #s 26, 47, 48

(26) Graph curve w/ parametric eq'ns

$x = \cos t,\ y = \sin t,\ z = \sin 5t$ & curvature at $(1,0,0)$

$\bar{r}' = \langle -\sin t, +\cos t, 5\cos 5t \rangle$, $-\sin t$, $\cos t$

$\times\ \bar{r}'' = \langle -\cos t, -\sin t, -5\sin 5t \rangle$, $-\cos t$, $-\sin t$

$\bar{r}' \times \bar{r}'' = \langle -5\sin 5t \cos t + 5\sin t \cos 5t,$
$-5\cos 5t \cos t - 5\sin 5t \sin t,\ \sin^2 t + \cos^2 t \rangle$

Just plug in $t=0$!

$$\frac{\|\bar{r}' \times \bar{r}''\|}{\|\bar{r}'\|^3} =$$

$$= \frac{\left(25\sin^2 5t\cos^2 t - 50\sin 5t \sin t \cos 5t \cos t + 25\sin^2 t\cos^2 5t + 25\cos^2 5t\cos^2 t + 50\sin 5t \sin t \cos 5t \cos t + 25\sin^2 5t \sin^2 t + 1\right)^{1/2}}{\left(\sqrt{\sin^2 t + \cos^2 t + 25\cos^2 5t}\right)^3}$$

$$= \frac{\left((25\sin^2 5t)(\sin^2 t + \cos^2 t) + (25\cos^2 5t)(\sin^2 t + \cos^2 t) + 1\right)^{\frac{1}{2}}}{\text{same}}$$

$$= \frac{\sqrt{25\sin^2 5t + 25\cos^2 5t + 1}}{(25\cos^2 5t)^{3/2}} = \frac{\sqrt{26}}{(5|\cos 5t|)^3}$$

$t = 0 \leftrightarrow \langle 1,0,0 \rangle \Rightarrow$

$$\kappa_{t=0} = \frac{\sqrt{26}}{125}$$

203 §13.3 #s 47, 48

#s 47-8 Find $\overline{T}, \overline{N}, \overline{B}$ @ the given point

47 $\overline{r} = \langle t^2, \frac{2}{3}t^3, t \rangle$, $P(1, \frac{2}{3}, 1) \longleftrightarrow t = 1$

$\overline{r}' = \langle 2t, 2t^2, 1 \rangle \Rightarrow$

$\|\overline{r}'\| = \sqrt{4t^2 + 4t^4 + 1}$

$\boxed{\overline{T} = (4t^2 + 4t^4 + 1)^{-\frac{1}{2}} \langle 2t, 2t^2, 1 \rangle}$

$\overline{T}' = -\frac{1}{2}(4t^2 + 4t^4 + 1)^{-\frac{3}{2}} (4t^2 + 4t^4 + 1) \langle 2t, 2t^2, 1 \rangle$

$+ (4t^2 + 4t^4 + 1)^{-\frac{1}{2}} \langle 2, 4t, 0 \rangle$

$= \dfrac{-\langle 4t^2, 4t^4, 1 \rangle}{2(4t^2 + 4t^4 + 1)^{3/2}} + \dfrac{4t^2 + 4t^4 + 1}{2(4t^4 + 4t^2 + 1)^{3/2}} \langle 2t, 2t^2, 1 \rangle$

$= \dfrac{\langle -4t^2, -4t^4, -1 \rangle + 2\langle 8t^3 + 8t^5 + 2t, 8t^4 + 8t^6 + 2t^2, 1 \rangle}{2(\quad)}$

$= \boxed{\dfrac{\langle 16t^5 + 16t^3 - 4t^2 + 4t, 16t^6 + 12t^4 + 2t^2, 1 \rangle}{2(4t^4 + 4t^2 + 1)^{3/2}} = \overline{T}'}$

$\|\overline{T}'\| = \dfrac{\left((16t^5 + 16t^3 - 4t^2 + 4t)^2 + (16t^6 + 12t^4 + 2t^2)^2 + 1\right)^{\frac{1}{2}}}{2(4t^4 + 4t^2 + 1)^{3/2}}$

Not the most efficient way!

$\Rightarrow \boxed{\overline{N} = \dfrac{\overline{T}'}{\|\overline{T}'\|}}$

$\overline{T}(1) = (4(1) + 4(1) + 1)^{-3/2} \langle 2(1), 2(1), 1 \rangle = \left(\frac{1}{9}\right)^{3/2} \langle 2, 2, 1 \rangle$

$= \boxed{\frac{1}{27} \langle 2, 2, 1 \rangle = \overline{T}(1)}$

203 § 13.3 cont'd

(47) $\bar{N}(1) = \dfrac{\langle 16+16-4 \cancel{+4}, 16+12+2, 1\rangle}{2(4+4+1)^{3/2}}$

$= \dfrac{\langle 32, 30, 1\rangle}{2(9)^{3/2}} = \dfrac{\langle 32,30,1\rangle}{54} = \bar{N}(1)$

$\bar{B}(1) = \bar{T}(1) \times \bar{N}(1) = \dfrac{1}{27}\cdot\dfrac{1}{54}\langle 2,2,1\rangle \times \langle 32,30,1\rangle$

Scratch:

$\langle 2, 2, 1\rangle, 2, 2$

$\times \langle 32, 30, 1\rangle, 32, 30$

$\langle -28, 30, -4\rangle$ $\Longrightarrow$ $\dfrac{1}{3^3}\cdot\dfrac{1}{3^3}\cdot\dfrac{1}{2}$

$$\boxed{\bar{N}(1) = \left(\frac{1}{2}\right)\left(\frac{1}{3^6}\right)\langle -28, 30, -4\rangle}$$

(48) $\bar{r}(t) = \langle \cos t, \sin t, \ln(\cos t)\rangle$, $\langle 1,0,0\rangle \longrightarrow t = 0$

$\bar{r}' = \left\langle -\sin t, \cos t, \dfrac{-\sin t}{\cos t}\right\rangle = \langle -\sin t, \cos t, -\tan t\rangle$

$\|\bar{r}'\| = \sqrt{\sin^2 t + \cos^2 t + \tan^2 t} = \sqrt{1+\tan^2 t}$

$= \sqrt{\sec^2 t} = |\sec t| = \sec t$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Longrightarrow \bar{T} = \dfrac{1}{\sec t}\langle -\sin t, \cos t, -\tan t\rangle$

$$\boxed{= (\cos t)\langle -\sin t, \cos t, -\tan t\rangle = \bar{T}}$$

203 §13.3 #48 cntd

(48) cntd

$\bar{T}' = (-\sin t)\langle -\sin t, \cos t, -\tan t\rangle$

$+ \cos t\langle -\cos t, -\sin t, -\sec^2 t\rangle$

$= \langle \sin^2 t - \cos^2 t, -2\sin t\cos t, \sin t\tan t - \cos t\sec^2 t\rangle$

Scratch $\sin^2 t - \cos^2 t = 1 - 2\cos^2 t$

$= 1 - 2\left(\frac{1+\cos 2t}{2}\right) = 1 - 1 - \cos(2t) = -\cos(2t)$

$-2\sin t\cos t = -\sin(2t)$

$\frac{\sin^2 t}{\cos t} - \frac{1}{\cos t} = \frac{-\cos^2 t}{\cos t} = -\cos t$

$\bar{T}' = \langle -\cos(2t), -\sin(2t), -\cos t\rangle$

$\Longrightarrow \|\bar{T}'\| = \sqrt{\cos^2(2t) + \sin^2(2t) + \cos^2 t}$

$= \sqrt{1 + \cos^2 t}$

$\bar{N} = (1+\cos^2 t)^{-1/2}\langle -\cos(2t), -\sin(2t), -\cos t\rangle$

$\bar{T}(0) = \frac{1}{1}\langle 0, 1, 0\rangle = \boxed{\langle 0, 1, 0\rangle = \bar{T}(0)}$

$\boxed{\bar{N}(0) = \frac{1}{\sqrt{2}}\langle -1, 0, -1\rangle}$

$\bar{T}(0) \quad \langle 0, 1, 0\rangle, 0, 1$

$\times \bar{N}(0) \quad \langle -1, 0, -1\rangle, -1, 0$

$\boxed{\frac{1}{\sqrt{2}}\langle -1, 0, 1\rangle = \bar{B}(0)}$

203 §13.3 #s 47, 48

#s 47-8 Find $\bar{T}, \bar{N}, \bar{B}$ @ $(1, \frac{2}{3}, 1) \to t=1$

47 $\bar{r} = \langle t^2, \frac{2}{3}t^3, t \rangle$

$\bar{r}' = \langle 2t, 2t^2, 1 \rangle$

$\|\bar{r}'\| = \sqrt{4t^2 + 4t^4 + 1}$

$\frac{\bar{r}'}{\|\bar{r}'\|} = \bar{T}(1) = \frac{\langle 2, 2, 1 \rangle}{\sqrt{4+4+1}} = \frac{\langle 2, 2, 1 \rangle}{\sqrt{9}} = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \rangle$
$= \bar{T}(1)$

$= \boxed{\frac{1}{3}\langle 2, 2, 1 \rangle = \bar{T}(1)}$

$\bar{N} = \frac{\bar{T}'}{\|\bar{r}'\|}$

$\bar{T}' = \frac{d}{dt}\left[(4t^2 + 4t^4 + 1)^{-\frac{1}{2}} \langle 2t, 2t^2, 1 \rangle\right]$

$= -\frac{1}{2}(4t^2 + 4t^4 + 1)^{-\frac{3}{2}}(8t + 16t^3) \langle 2t, 2t^2, 1 \rangle$

$+ (4t^2 + 4t^4 + 1)^{-\frac{1}{2}} \langle 2, 4t, 0 \rangle$

$\bar{T}'(0) = -\frac{1}{2}(4+4+1)^{-3/2}(8+16) \langle 2, 2, 1 \rangle$

$+ (9)^{-1/2} \langle 2, 4, 0 \rangle = -\frac{1}{2}\left(\frac{1}{9}\right)^{3/2}(24) \langle 2, 2, 1 \rangle$

$+ \frac{1}{3}\langle 2, 4, 0 \rangle = -12\left(\frac{1}{27}\right)\langle 2, 2, 1 \rangle$

$+ \langle \frac{2}{3}, \frac{4}{3}, 0 \rangle$

203 §13.3 #47, 48

47 cont'd $\qquad \frac{12}{27} = \frac{4}{9}$

$$= -\frac{4}{9}\langle 2, 2, 1\rangle + \langle \tfrac{2}{3}, \tfrac{4}{3}, 0\rangle$$

$$= \langle \tfrac{8}{9}, \tfrac{8}{9}, \tfrac{4}{9}\rangle + \langle \tfrac{6}{9}, \tfrac{12}{9}, 0\rangle$$

$$= \langle \tfrac{14}{9}, \tfrac{20}{9}, \tfrac{4}{9}\rangle = T'(1)$$

$$\Rightarrow \|T'(1)\| = \sqrt{\frac{16 + 400 + 16}{81}} = \sqrt{\frac{432}{81}}$$

$$= \frac{8}{9}\sqrt{7}$$

$$\begin{array}{r|l} 2 & 432 \\ 2 & 216 \\ 2 & 108 \\ 2 & 56 \\ 2 & 28 \\ 2 & 14 \\ & 7 \end{array}$$