Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

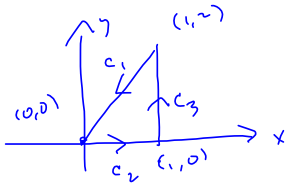
In the sequel, we might think of this as being somewhat "curly." But for now, "curl" is just something to do with hair.

Green's Theorem should be regarded as the counterpart of the Fundamental Theorem of Calculus for double integrals. Compare Equation 1 with the statement of the Fundamental Theorem of Calculus, Part 2, in the following equation:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Sometimes we'd rather do a line integral. Sometimes we'd rather do a double integral.

#3 $\int_C xy\,dx + x^2y^3\,dy$ C: triangle $(0,0)$, $(1,0)$, $(1,2)$



Need to GIVE it an orientation:

c_1: $y=2x$, $x=1$ to 0

c_2: $y=0$, $x=0$ to 1

c_3 $x=1$, $y=0$ to $y=2$

1–4 Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem.

3. $\oint_C xy\,dx + x^2y^3\,dy$,
C is the triangle with vertices (0, 0), (1, 0), and (1, 2)

c_2 Old way: $\vec r(t) = (1-t)\langle 0,0\rangle + t\langle 1,0\rangle$
$= \langle t, 0\rangle \quad 0 \le t \le 1$

$\int_{c_2} xy\,dx + x^2y^3\,dy$
$= \int_0^1 t\cdot 0\,dx + t^2\cdot 0^3\,dy = 0$

c_1 $(1,2)$ to $(0,0)$
$(1-t)\langle 1,2\rangle + t\langle 0,0\rangle = \langle 1-t, 2-2t\rangle$

$x = 1-t \quad y = 2-2t$
$dx = -dt \quad dy = -2dt$

$\int_{c_1} \sim = \int_0^1 (1-t)(2-2t)(-dt) + (1-t)^2(2-2t)^3\cdot(-2\,dt)$

c_3: $x=1$, $y=0$ to 2

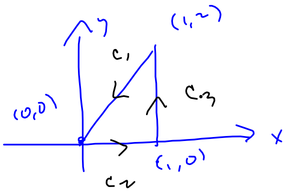
$(1,0)$ to $(1,2)$
$(1-t)\langle 1,0\rangle + t\langle 1,2\rangle$
$= \langle 1-t, 0\rangle + \langle t, 2t\rangle$
$= \langle 1, 2t\rangle \quad 0 \le t \le 1$

$y = 2t$
$dy = 2\,dt$

$\int_{c_3} \sim =$
$\int_0^1 1\cdot 2t\cdot 0 + 1^2\cdot(2t)^3\cdot 2\,dt$

1–4 Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem.

3. $\oint_C xy\,dx + x^2y^3\,dy$,
C is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$



C_1: $y = 2x$ $x = 1$ to $x = 0$

C_2 $y = 0$, $x = 0$ to $x = 1$

C_3 $x = 1$ $y = 0$ to 2

$y = 2x$
$dy = 2\,dx$

$$\int_{C_1} xy\,dx + x^2y^3\,dy = \int_1^0 x \cdot 2x\,dx + (x)^2(2x)^3(2dx)$$

$$= \int_1^0 2x^2\,dx + \int_1^0 16x^5\,dx$$

$$\int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA = \int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx$$

$P = xy \Rightarrow P_y = x$
$Q = x^2y^3 \Rightarrow Q_x = 2xy^3$

C_2: $y = 0$ $0 \le x \le 1$

$$\int_{C_2} P\,dx + Q\,dy = \int_C xy\,dx + x^2y^3\,dy$$

$$= \int_0^1 x \cdot 0 \cdot dx + x^2 \cdot \cancel{0}^3 \cdot 0 = 0$$

$y = 0$ to 2

C_3: $x = 1$, $0 \le y \le 2$

$x = 1 \Rightarrow$
$dx = 0$

$$\int_0^2 1y \cdot 0 + \int_0^2 1^2 y^3\,dy \quad \left[\frac{y^4}{4}\right]_0^2 = 4$$

$$\oint_C xy\,dx + x^2y^3\,dy = I_1 + I_2 + I_3 = -\frac{10}{3} + \frac{12}{3} = \frac{2}{3}$$

If $Q_x - P_y = 1$, Then $\iint (Q_x - P_y)\,dA = \iint dA = \text{Area} = \oint_C P\,dx + Q\,dy$

$P(x, y) = 0$ $\quad$ $P(x, y) = -y$ $\quad$ $P(x, y) = -\frac{1}{2}y$

$Q(x, y) = x$ $\quad$ $Q(x, y) = 0$ $\quad$ $Q(x, y) = \frac{1}{2}x$

1 $\quad$ 2 $\quad$ 3

Then Area $= \underset{1}{\int_C x\,dy} = \underset{2}{-\int_C y\,dx} = \underset{3}{\frac{1}{2}\int x\,dy - y\,dx}$

EXAMPLE 3 Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

use 3.

$$\iint_D 1\,dA = \iint_D (Q_x - P_y)\,dA = \int_C P\,dx + Q\,dy$$

$$= \int_C -\tfrac{1}{2}y\,dx + \tfrac{1}{2}x\,dy$$

$x = a\cos\theta$ $\quad$ $0 \le \theta \le 2\pi$

$y = b\sin\theta$

$dx = -a\sin\theta\,d\theta$

$dy = b\cos\theta\,d\theta$

$$= \frac{1}{2}\int_0^{2\pi} -b\sin\theta(-a\sin\theta\,d\theta) + a\cos\theta\, b\cos\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,d\theta = \frac{1}{2}\cdot 2\pi ab = \pi ab.$$

Our Green's Thm is only for simple regions.
It extends naturally to the sum of simple regions.
$\rightarrow$ union

It can be extended to regions w/ holes

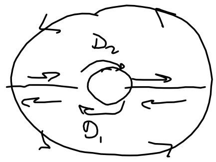


Fig 10.

$$\iint_D = \iint_{D_1} + \iint_{D_2}$$

D_1 & D_2 have no holes

(Just don't be surprised if what you think are conservative vector fields aren't.

§16.5

Recall: $\cancel{\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial}{\partial z} \rangle}$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \text{gradient}$$

$$\bar{F} = \langle P, Q, R \rangle$$

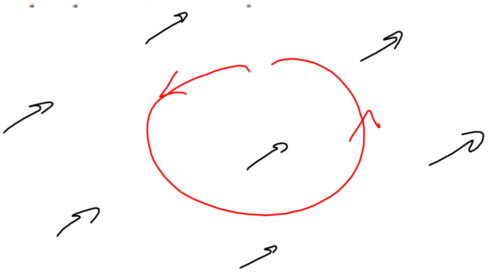
Divergence
$$\nabla \cdot \bar{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle$$
$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \operatorname{div}(\bar{F}) = \text{divergence.}$$

Curl: $\nabla \times \bar{F}$

$$\begin{array}{r} \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle, \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \\ \times \; \langle P, Q, R \rangle, P, Q \\ \hline \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle \end{array}$$

12
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k} \, dA$$

Equation 12 expresses the line integral of the tangential component of **F** along C as the double integral of the vertical component of curl **F** over the region D enclosed by C. We now derive a similar formula involving the *normal* component of **F**.

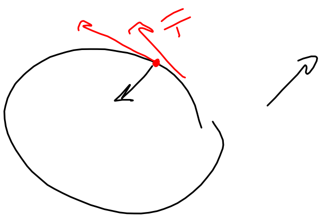


$$\bar{T} = \frac{1}{\sqrt{x'^2+y'^2}} \langle x', y' \rangle$$

$$\bar{N} = \frac{1}{\sqrt{x'^2+y'^2}} \langle -y', x' \rangle$$ is inward normal from previous chapter.

$$\bar{T} \cdot \bar{N} = \frac{1}{x'^2+y'^2} (-x'y' + x'y') = 0$$

$$\bar{n} = \text{outward normal} = -\bar{N} = \frac{1}{\|\bar{r}'(t)\|} \langle y', -x' \rangle$$



$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_a^b (\mathbf{F}\cdot\mathbf{n})(t)\,|\mathbf{r}'(t)|\,dt$$

$$= \int_a^b \left[\frac{P(x(t), y(t))\,y'(t)}{|\mathbf{r}'(t)|} - \frac{Q(x(t), y(t))\,x'(t)}{|\mathbf{r}'(t)|}\right]|\mathbf{r}'(t)|\,dt$$

$$= \int_a^b P(x(t), y(t))\,y'(t)\,dt - Q(x(t), y(t))\,x'(t)\,dt$$

5, 12, 19
6, 13, 20

$$= \int_C P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA$$

$$\int P\,dx + Q\,dy = \iint (Q_x - P_y)\,dA$$

$$\int (-Q)\,dx + P\,dy = \iint (P_x - (-Q_y))\,dA = \iint (P_x + Q_y)\,dA$$

13 $$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \operatorname{div}\mathbf{F}(x, y)\,dA$$

This version says that the line integral of the normal component of **F** along C is equal to the double integral of the divergence of **F** over the region D enclosed by C.