PLEASE NOTE: I UPLOADED A NEW 16.3 ASSIGNMENT SHEET.

Basically, I added #s 15, 16. I'd also like you to give #36 some thought.

FTC II Integral on the interval reduces to evaluation of antiderivative at the endpoints.

The endpoints of an interval comprise its boundary.

We will see the 2-D version of this, where the boundary is a closed curve and the interior is an open set in the plane.

$$\boxed{1} \quad \int_a^b F'(x)\,dx = F(b) - F(a)$$

See Conservation of Energy

We also called Equation 1 the Net Change Theorem: The integral of a rate of change is the net change.

$$\int \langle P(x,y), Q(x,y)\rangle \cdot \underbrace{\langle x', y'\rangle\, dt}_{F'(t)\,dt} \quad \int P\, x'(t)\,dt + \int Q\, y'(t)\,dt \quad \int P\,dx + Q\,dy$$

$\boxed{2}$ **Theorem** Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

This is why we LOVE conservative vector fields!

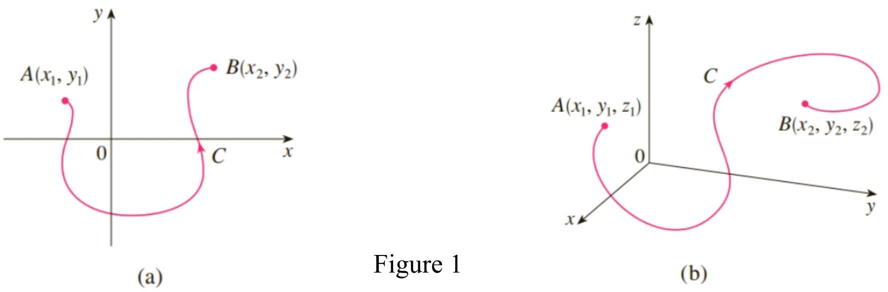


Figure 1

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2) - f(x_1, y_1) \qquad \int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

PROOF OF THEOREM 2

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt$$

$$= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\, dt \qquad \text{(by the Chain Rule)}$$

$$= f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

$\nabla f = \bar{F} = \langle f_x, f_y, f_z \rangle$

$\langle x'(t), y'(t), z'(t) \rangle$

EXAMPLE 1 Find the work done by the gravitational field

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x} = \frac{-mMG}{(x^2+y^2+z^2)^{3/2}}\langle x, y, z \rangle$$

Let's give the gravity model a better treatment than last time.

in moving a particle with mass m from the point (3, 4, 12) to the point (2, 2, 0) along a piecewise-smooth curve C. (See Example 16.1.4.)

$$\bar{F} = mMG \left\langle \frac{-x}{\|\bar{x}\|^3}, \frac{-y}{\|\bar{x}\|^3}, \frac{-z}{\|\bar{x}\|^3} \right\rangle$$

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

Plug in $f(2,2,0) - f(3,4,12)$

The upshot: Line integrals of conservative vector fields are *independent of path*!

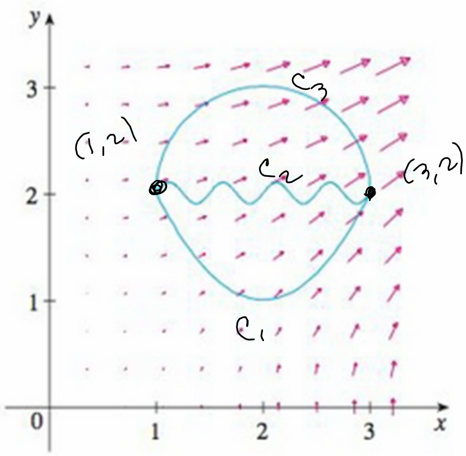
As a consequence, when line integrals are independent of path, then the line integral over a closed curve will be zero, because the endpoints are the same!

3 Theorem $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

11. The figure shows the vector field $\mathbf{F}(x, y) = \langle 2xy, x^2 \rangle$ and three curves that start at (1, 2) and end at (3, 2).

(a) Explain why $\int_C \mathbf{F} \cdot d\mathbf{r}$ has the same value for all three curves.

(b) What is this common value?



If $\bar{F}$ is conservative, then (a) is done.

If $\bar{F} = \nabla f$ for some f...

$\Rightarrow f_x = 2xy$

$f_y = x^2$

$f_x = 2xy \Rightarrow f = x^2 y + g(y)$

$\&$ $f_y = x^2 + g'(y) = x^2$

$\Rightarrow g'(y) = 0 \Rightarrow$

$g(y) \equiv K$

$\Rightarrow f(x,y) = x^2 y + K \quad \forall K \in \mathbb{R}$,

in particular $K = 0$.

$f(x,y) = x^2 y$ works.

(a) is done.

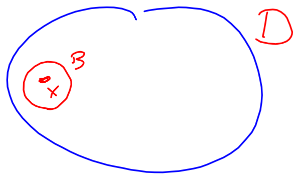
$\nabla f = \langle 2xy, x^2 \rangle$

a

$$\int_{C_x} \bar{F} \cdot d\bar{r} = f(3,2) - f(1,2)$$

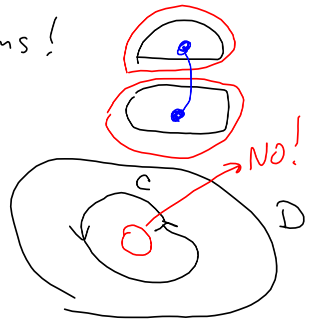
$$= 3^2 \cdot 2 - 1^2 \cdot 2 = 8 \cdot 2 = 16 = \int_{C_x} \bar{F} \cdot d\bar{r}$$

Open: $x \in D \rightarrow \exists$ open ball $B \subseteq D$



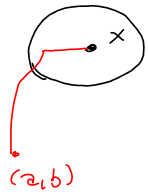
Connected: No separations!

Simply Connected
connected with no holes!



If D has a hole, we can find a closed curve in D that encloses points NOT in D.

4 Theorem Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.



5 Theorem If $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

We would like the (a) *converse* of this to also hold. For that, we need *simply* connected.

6 Theorem Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

This is a *quick* way to determine if a field is conservative!

See examples 2 and 3.

EXAMPLE 5 If $\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + (2xy + e^{3z})\,\mathbf{j} + 3ye^{3z}\,\mathbf{k}$, find a function f such that $\nabla f = \mathbf{F}$.

$$\bar{F} = \langle y^2,\ 2xy + e^{3z},\ 3ye^{3z} \rangle$$

$$f_x = y^2 \Rightarrow f = xy^2 + g(y,z)$$

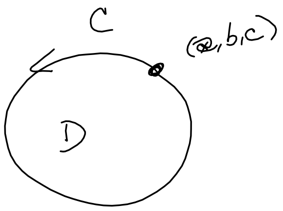
$$f_y = 2xy + e^{3z} = 2xy + g_y(y,z) \Rightarrow g_y(y,z) = e^{3z}$$

$$\Rightarrow f = xy^2 + ye^{3z} + h(z)$$

$$f_z = 3ye^{3z} = 3ye^{3z} + h'(z) \Rightarrow h'(z) = 0 \Rightarrow h(z) = K \equiv 0$$

$$f(x,y,z) = x \cdot y^2 + ye^{3z}$$

$$\nabla f = \langle y^2,\ 2xy + e^{3z},\ 3ye^{3z} \rangle$$



$$\int_C \bar{F} \cdot d\bar{r} =$$

$$\int_C \bar{F} \cdot d\bar{r}$$

$$= \int_C \nabla f \cdot d\bar{r} = f(a,b,c) - f(a,b,c)$$

$\bar{F}$ conservative, D simply connected.

Section 16.4 Sneak Preview

Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

In the sequel, we might think of this as being somewhat "curly." But for now, "curl" is just something to do with hair.

Green's Theorem should be regarded as the counterpart of the Fundamental Theorem of Calculus for double integrals. Compare Equation 1 with the statement of the Fundamental Theorem of Calculus, Part 2, in the following equation:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$