2. (10 pts) Write as many of the 5 other iterated integrals that are equivalent to the iterated integral $\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)\,dz\,dy\,dx$ as you can find. 2 points apiece.

$$\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{3}} \sin\phi\cos\theta\; d\phi\, d\theta$$

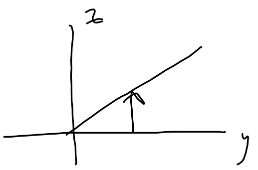
I may owe some points

$$=\int_0^{\frac{\pi}{4}}\cos\theta\, d\theta\int_0^{\frac{\pi}{3}}\sin\phi\, d\phi$$

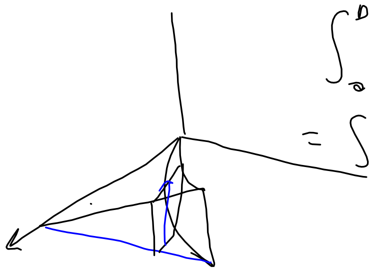
This is T11.

Check Fubini

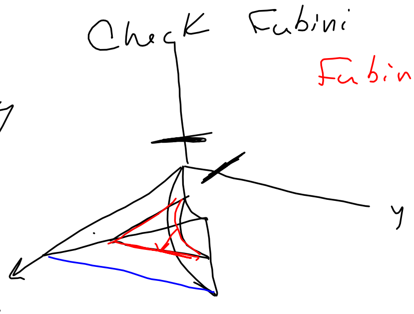
Fubini is T10.



$$\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$$



$$\int_0^1\int_{\sqrt{y}}^1\int_0^y dz\,dx\,dy$$



$$\int_0^1\int_{\sqrt{z}}^1\int_z^{x^2} dy\,dx\,dz$$

$$\int_0^1\int_z^1\int_{\sqrt{y}}^1 dx\,dy\,dz$$

1 DEFINITION Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

2 DEFINITION Let E be a subset of $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

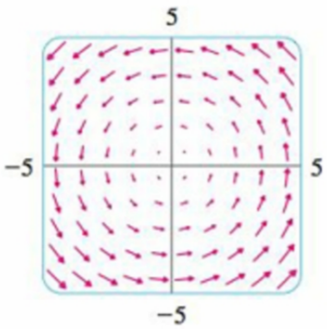


FIGURE 6
$\mathbf{F}(x, y) = \langle -y, x \rangle$

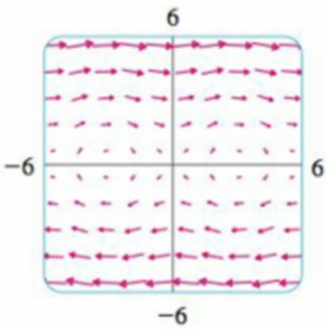


FIGURE 7
$\mathbf{F}(x, y) = \langle y, \sin x \rangle$

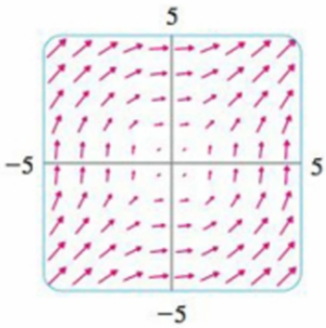


FIGURE 8
$\mathbf{F}(x, y) = \langle \ln(1 + y^2), \ln(1 + x^2) \rangle$

$\bar{x} = \langle x, y, z \rangle$

$$\mathbf{F}(x, y, z) = \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{k}$$

$$\left\langle \left(\frac{-mMG}{\|\bar{x}\|^2}\right)\frac{x}{\|\bar{x}\|},\ \left(\frac{-mMG}{\|\bar{x}\|^2}\right)\frac{y}{\|\bar{x}\|},\ \ldots \right\rangle$$

FIGURE 14
Gravitational force field

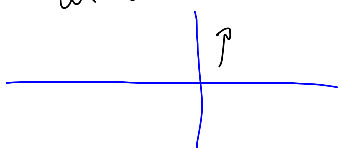
$\frac{3}{2} = 1 + \frac{1}{2}$

$$|\mathbf{F}| = \frac{mMG}{r^2} \qquad -\frac{\mathbf{x}}{|\mathbf{x}|} \qquad \mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\,\mathbf{x}$$

Euler's Method

$y' = 5x + 2y$

we saw this.



EXAMPLE 6 Find the gradient vector field of $f(x, y) = x^2y - y^3$.

A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation f is called a **potential function** for $\mathbf{F}$.

Not all vector fields are conservative, but such fields do arise frequently in physics. For example, the gravitational field $\mathbf{F}$ in Example 4 is conservative because if we define

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

This is nice and smooth, BUT it blows up @ (0,0,0)

$\nabla f = -mMG <$

$$\frac{\partial}{\partial x}\left[(x^2+y^2+z^2)^{-\frac{1}{2}}\right] = -\frac{1}{2}\left[x^2+y^2+z^2\right]^{-\frac{3}{2}}[2x]$$

$$= \frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

The $\vec{F}$ on previous page is Conservative Vector Field because it's the gradient of a function $f(x,y,z)$.

Section 16.2 - Line Integrals!!!

2 DEFINITION If f is defined on a smooth curve C given by Equations 1, then the **line integral of f along C** is

$$\int_C f(x, y)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

if this limit exists.

The increment of arc length is the same as before:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \left|\bar{r}'(t)\right| dt$$

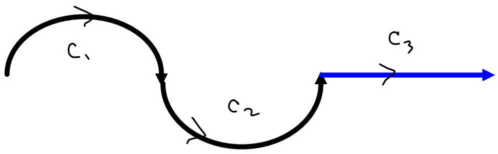
$\bar{r} = \bar{r}(t)$ is parametrized wrt "t."

Recall the arc-length integral:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b ds$$

If f is (piecewise-) continuous, we can integrate along (the pieces of) C :

3 $$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$



$$= \sum_{k=1}^{3} \int_{C_k} f(x,y)\, ds$$

line integrals of f along C with respect to x and y:

$$\int_C f(x, y)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta x_i$$

$$\int_C f(x, y)\, dy = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta y_i$$

$$\int_C f(x, y)\, dx = \int_a^b f\big(x(t), y(t)\big)\, x'(t)\, dt$$

$$\int_C f(x, y)\, dy = \int_a^b f\big(x(t), y(t)\big)\, y'(t)\, dt$$

15. $\int_C (x + yz)\,dx + 2x\,dy + xyz\,dz$, C consists of line segments from $(1, 0, 1)$ to $(2, 3, 1)$ and from $(2, 3, 1)$ to $(2, 5, 2)$

$\bar{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

$$\int_C \bar{F} \cdot d\bar{r} = \int_a^b \langle P, Q, R \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle \, dt$$

$d\bar{r} = \bar{r}'(t)\,dt$ $\qquad \bar{r} = \langle x(t), y(t), z(t) \rangle$

along the curve C.

$f(x,y) \longrightarrow f(x(r,\theta), y(r,\theta))$

$$= \int_a^b \left(P\,x'(t) + Q\,y'(t) + R\,z'(t) \right) dt$$

$$= \int_a^b P\,x'(t)\,dt + \int_a^b Q\,y'(t)\,dt + \int_a^b R\,z'(t)\,dt$$

$$= \int_a^b P\,dx + \int_a^b Q\,dy + \int_a^b R\,dz$$

$= \text{Book Says} = \int_a^b P\,dx + Q\,dy + R\,dz$ Ugh!

15. $\int_C (x + yz)\,dx + 2x\,dy + xyz\,dz$, C consists of line segments from (1, 0, 1) to (2, 3, 1) and from (2, 3, 1) to (2, 5, 2)

$(1-t)\langle 1,0,1\rangle + t\langle 2,3,1\rangle$

$= \langle 1-t+2t,\ 0+3t,\ 1-t+t\rangle$

$= \boxed{\langle t+1,\ 3t,\ 1\rangle = \bar{r}_1(t)}$

$\bar{r}_2(t) = (1-t)\langle 2,3,1\rangle + t\langle 2,5,2\rangle = \langle 2-2t+2t,\ 3-3t+5t,\ 1-t+2t\rangle$

$= \underline{\langle 2,\ 3+2t,\ 1+t\rangle = \bar{r}_2(t)}$

$$\int_C (x+yz)\,dx + 2x\,dy + xyz\,dz$$

$$= \int_{C_1} + \int_{C_2} \qquad = \underline{\langle 2,\ 3+2t,\ 1+t\rangle = \bar{r}_2(t)}$$

I_1: $= \langle t+1,\ 3t,\ 1\rangle = \bar{r}_1(t)$

$$\int_{C_1} : \quad \int_0^1 \big(t+1+(3t)\big)\cdot 1\,dt + \int_0^1 2(t+1)\cdot 3\,dt + \cancel{\int_0^1 (t+1)(3t)\cdot 0\,dt}$$

LINE INTEGRALS OF VECTOR FIELDS

Recall the connection between dot product of vectors and cosine of the angle between them. Generally the dot product gives us a measure of "how large a shadow" the one vector casts on the other vector. When vectors are orthogonal (perpendicular), the dot product is ZERO. When they're in the same direction (at least partially), the dot product is positive, and when the angle between them is greater than 90 degrees, the dot product is *negative*.

Work in one dimension. Constant force F, distance D.

$$W = FD$$

Variable force:

$$\int_a^b F(x)\,dx$$ which simplifies to the above when F is constant.

Work in 2 or 3 dimensions.

The **work** W done by the force field $\mathbf{F}$

$$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z)\, ds = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

$\mathbf{T}$ gives us a unit vector in the direction of the curve

$$= \int_a^b \left[\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right] |\mathbf{r}'(t)|\, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_a^b \vec{F} \cdot d\vec{F}$$

13 DEFINITION Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

$$= \int_a^b \left[P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t)\right] dt$$

$$= \int_C P\,dx + Q\,dy + R\,dz \qquad \text{where } \mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k} = \langle P, Q, R \rangle$$

23–26 Use a calculator or CAS to evaluate the line integral correct to four decimal places. Missing the "*dt*" in this discussion (in Videos).

23. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = xy\,\mathbf{i} + \sin y\,\mathbf{j}$ and $\mathbf{r}(t) = e^{t}\,\mathbf{i} + e^{-t^2}\,\mathbf{j}$, $1 \leqslant t \leqslant 2$

$\bar{r} = \langle e^t, e^{-t^2} \rangle$

$\bar{F} = \langle xy, \sin y \rangle$

$\bar{r} = \langle e^t, e^{-t^2} \rangle$

$\bar{F} = \langle xy, \sin y \rangle$

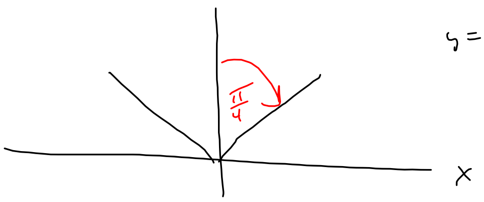
$\bar{F} = \langle e^t e^{-t^2}, \sin(e^{-t^2}) \rangle$

$\bar{r}' = \langle e^t, -2te^{-t^2} \rangle$

$e^{2t-t^2} - 2te^{-t^2}\sin(e^{-t^2})$

$$\int_1^2 \left(e^{-t^2+2t} - 2te^{-t^2}\sin(e^{-t^2})\right) dt$$

$z = \sqrt{x^2+y^2}$



$y=0$

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho\cos\phi = \sqrt{\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta}$

$= \sqrt{\rho^2 \sin^2\phi} = |\rho \sin\phi| = \rho\sin\phi$

in 1st oct.

$\Rightarrow \cos\phi = \sin\phi \Rightarrow$

$\phi = \frac{\pi}{4}$