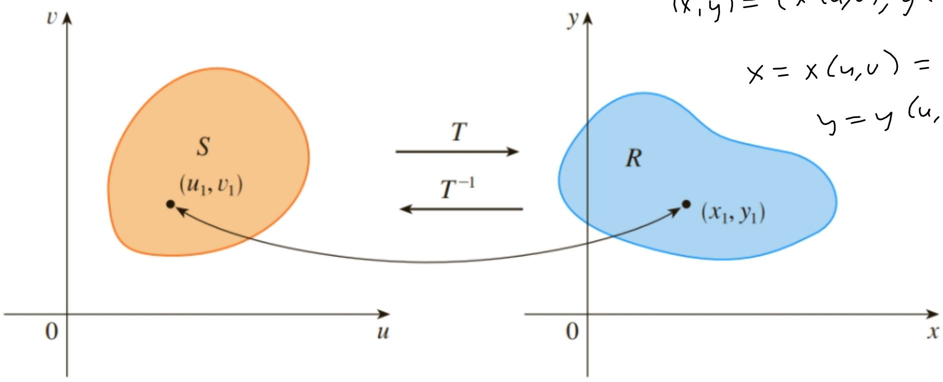
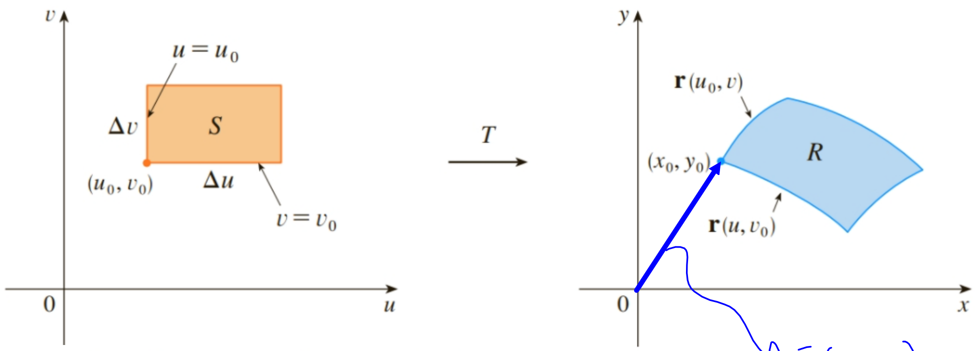
$(x, y) = (x(u,v), y(u,v))$

$x = x(u,v) = g(u,v)$

$y = y(u,v) = h(u,v)$





$\bar{r}(u_0, v_0)$

$= \langle x(u_0, v_0), y(u_0, v_0) \rangle$

$$\mathbf{r}(u, v) = g(u, v)\,\mathbf{i} + h(u, v)\,\mathbf{j}$$

$= \langle g(u,v), h(u,v) \rangle$

$$\mathbf{r}_u = g_u(u_0, v_0)\,\mathbf{i} + h_u(u_0, v_0)\,\mathbf{j} = \left.\frac{\partial x}{\partial u}\right|_{(u,v)=(u_0,v_0)}\mathbf{i} + \frac{\partial y}{\partial u}\,\mathbf{j}$$

$\parallel$

$F(u_0, v_0) = \langle g_u(u_0, v_0), h_u(u_0, v_0) \rangle$

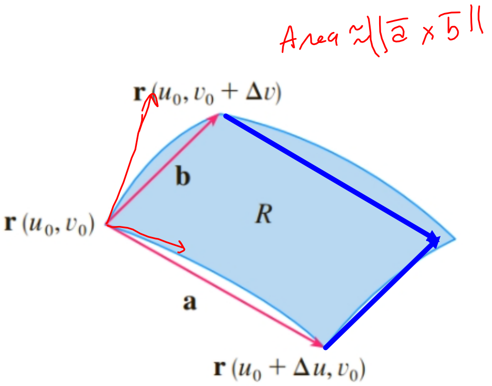


FIGURE 4

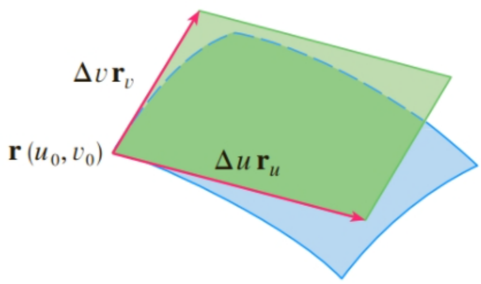


FIGURE 5

$$\mathbf{a} = \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \qquad \mathbf{b} = \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0)$$

$$\mathbf{r}_u = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u} \qquad \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \Delta v\, \mathbf{r}_v$$

$$\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \Delta u\, \mathbf{r}_u \approx \bar{a}\,! \qquad \bar{b} \approx \bar{r}_v \Delta v$$

$$\bar{a} \times \bar{b} \approx (\bar{r}_u \Delta u) \times (\bar{r}_v \Delta v) = (\bar{r}_u \times \bar{r}_v)\,\Delta u \Delta v$$

$$|(\Delta u\, \mathbf{r}_u) \times (\Delta v\, \mathbf{r}_v)| = |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u\, \Delta v$$

FACT : The determinant of the transpose =
The " " " original matrix

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow |A| = ad - bc$ This is true

$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \Rightarrow |A| = ad - bc$ for $3\times3, 4\times4, \ldots$

$$\boxed{\mathbf{r}_u \times \mathbf{r}_v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}$$

$$\langle x_u, y_u, 0\rangle,\ x_u, y_u$$
$$\times\ \langle x_v, y_v, 0\rangle,\ x_v, y_v$$
$$\langle 0, 0, x_u y_v - x_v y_u\rangle$$

7 Definition The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

9 **Change of Variables in a Double Integral** Suppose that T is a C^1 transformation whose Jacobian is nonzero and that T maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

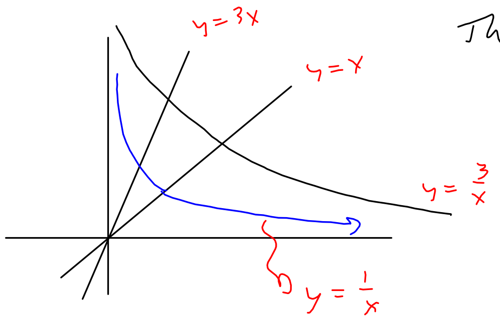
$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

$\|\bar{r}_u \times \bar{r}_v\|$

$$\iint_R xy\, dA$$

$y = \frac{1}{x}$ $\quad y = \frac{3}{x}$

$R: \quad y = x, \ y = 3x, \ xy = 1, \ xy = 3$



Their Suggestion

$y = v$

$x = \frac{u}{v}$

$\bar{r} = \langle \frac{u}{v}, v, 0 \rangle$

$\bar{r}_u = \langle \frac{1}{v}, 0, 0 \rangle, \frac{1}{v}, 0$

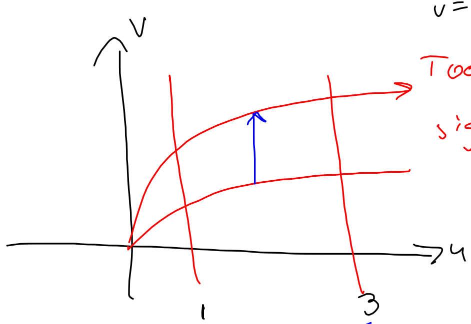
$\times \quad \bar{r}_v = \langle -\frac{u}{v^2}, 1, 0 \rangle, -\frac{u}{v^2}, 1$

$\langle 0, 0, \frac{1}{v} \rangle$

$$\iint_D \left(\frac{u}{v} v\right)\left(\frac{1}{v}\right) du\, dv \qquad \& \ \|\bar{r}_u \times \bar{r}_v\| = \sqrt{\frac{1}{v^2}} = \frac{1}{v}$$

what's the domain in the uv-plane?

$xy = 1$	$xy = 3$	$y = x$	$y = 3x$
$\frac{u}{v} \cdot v = 1$	$\frac{u}{v} \cdot v = 3$	$v = \frac{u}{v}$	$v = \frac{3u}{v}$
$u = 1$	$u = 3$	$u = v^2$	$v^2 = 3u$
		$v = \sqrt{u}$	$v = \sqrt{3u}$



Took the positive b/c everything in sight is QI

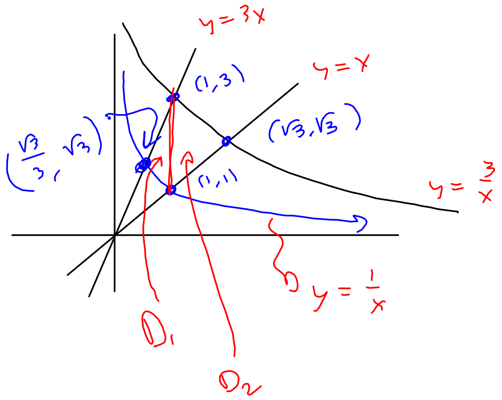
Keenan sez $\int_1^3$

$$\int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} \frac{u}{v}\, dv\, du = \int_1^3 u \left[\ln v\right]_{\sqrt{u}}^{\sqrt{3u}} du$$

$$= \int_1^3 u \left[\ln(\sqrt{3u}) - \ln(\sqrt{u})\right] du = \frac{1}{2} \int_1^3 u \ln 3 \, du$$

$$= \frac{\ln 3}{2} \left[\frac{u^2}{2}\right]_1^3 = \frac{\ln 3}{2} \left[\frac{9}{2} - \frac{1}{2}\right] = 2 \ln 3$$

$\ln\sqrt{3} + \ln\sqrt{u} - \ln\sqrt{u} = \ln\sqrt{3} = \frac{1}{2}\ln 3$



$$\int_{\frac{1}{\sqrt{3}}}^{1}\int_{\frac{1}{x}}^{3x} xy\,dy\,dx + \int_{1}^{\sqrt{3}}\int_{x}^{\frac{3}{x}} xy\,dy\,dx$$

w/o change of variables.

3 Variables?

Increment of volume for this picture, using Jacobian (Change of variables)

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

Find volume w/ scalar triple product.

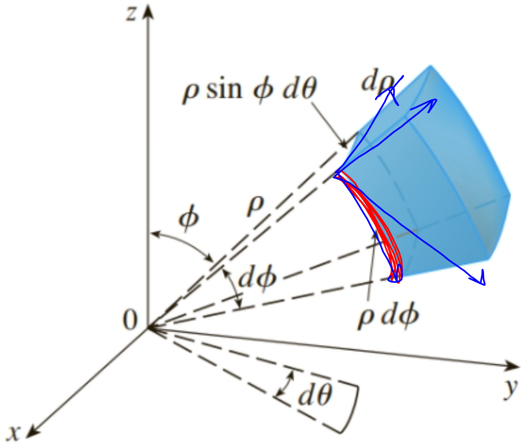


FIGURE 8

Volume element in spherical coordinates: $dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

$\bar{r} = \langle \rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi \rangle$

$\bar{r}_\rho = \langle \sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi \rangle$

$\bar{r}_\theta = \langle -\rho \sin\phi \sin\theta, \rho \sin\phi \cos\theta, 0 \rangle$

$\bar{r}_\phi = \langle \rho \cos\phi \cos\theta, \rho \cos\phi \sin\theta, -\rho \sin\phi \rangle$

$\bar{r}_\phi \cdot (\bar{r}_\rho \times \bar{r}_\theta)$ b/c it looks like easiest cross product.

As before:

$$\text{Volume} = (\bar{r}_\rho \Delta\rho) \cdot \left((\bar{r}_\theta \Delta\theta) \times (\bar{r}_\phi \Delta\phi) \right)$$

$$= \bar{r}_\rho \cdot (\bar{r}_\theta \times \bar{r}_\phi) \Delta\rho \Delta\theta \Delta\phi \longrightarrow \underbrace{\bar{r}_\rho \cdot (\bar{r}_\theta \times \bar{r}_\phi) d\rho d\theta d\phi}_{\text{Increment of Volume}}$$

Book says:

$$\left| \left[\bar{r}_u \,|\, \bar{r}_v \,|\, \bar{r}_w \right] \right|$$

I do

$$\begin{vmatrix} \bar{r}_u \\ \bar{r}_v \\ \bar{r}_w \end{vmatrix}$$

which is the transpose.

my way

$$\begin{vmatrix} x_u, y_u, z_u \\ x_v, y_v, z_v \\ x_w, y_w, z_w \end{vmatrix} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$= \langle x_u, y_u, z_u \rangle \cdot \left(\langle x_v, y_v, z_v \rangle \times \langle x_w, y_w, z_w \rangle \right)$$

Higher dimension? Hypervolume?

If higher Dimensions works same

$$\begin{pmatrix} \bar{r}_s \\ \bar{r}_t \\ \bar{r}_u \\ \bar{r}_v \end{pmatrix}$$

where $\bar{r}_s = \langle x_s, y_s, z_s, w_s \rangle$

$$\begin{vmatrix} x_s & y_s & z_s & w_s \\ x_t & y_t & z_t & w_t \end{vmatrix}$$