15.7 - Triple Integrals in Cylindrical Coordinates

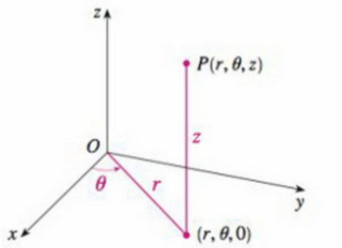


FIGURE 2
The cylindrical coordinates of a point

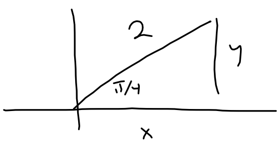
$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

1–2 Plot the point whose cylindrical coordinates are given. Then find the rectangular coordinates of the point.

1. (a) $(2, \pi/4, 1)$ (b) $(4, -\pi/3, 5)$

$r = 2, \ \theta = \frac{\pi}{4}$



$y = 2\left(\frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{2}\sqrt{2}} = \sqrt{2}$

$x = \frac{2}{\sqrt{2}} = \sqrt{2}$

$z = 1$

EVALUATING TRIPLE INTEGRALS WITH CYLINDRICAL COORDINATES

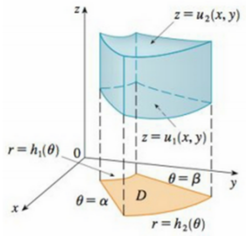


FIGURE 6

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leq z \leq u_2(x, y)\}$$

where D is given in polar coordinates by

TYPE 1, Always.

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$$

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

Recall from Section 15.4 on Double Integrals in Polar Coordinates:

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$$

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

$$\int_{u_1(x,y)}^{u_2(x,y)} \sim\, dz = G(x, y)$$

OR

$$G(r\cos\theta, r\sin\theta)$$

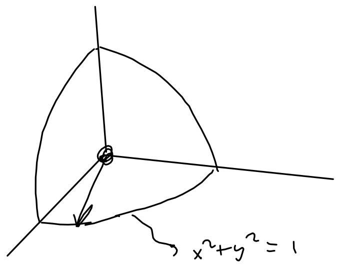
Combine to obtain the triple integral in CYLINDRICAL coordinate.

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

18. Evaluate $\iiint_E (x^3 + xy^2)\, dV$, where E is the solid in the first octant that lies beneath the paraboloid $z = 1 - x^2 - y^2$.

$= 1 - (x^2+y^2)$

$= 1 - r^2$



$$\int_0^{1-(x^2+y^2)}$$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} dy\, dx$$

$0 \le r \le 1$

$0 \le \theta \le \frac{\pi}{2}$

$r\, dz\, dr\, d\theta$

$$\int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{1-r^2} (x^3 + xy^2)\, dz\, r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{1-r^2} (r^3\cos^3\theta + r\cos\theta\, r^2\sin^2\theta)\, r\, dz\, dr\, d\theta$$

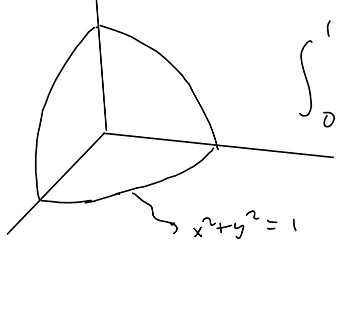
$$= \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{1-r^2} r^4 \cos\theta\, dz\, dr\, d\theta = \int_0^{\frac{\pi}{2}} \int_0^1 \left[r^4 \cos\theta\, z \right]_0^{1-r^2} dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 r^4 \cos\theta \left[1 - r^2\right] dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos\theta\, d\theta \int_0^1 (r^4 - r^6)\, dr$$

$$= \left(\sin\tfrac{\pi}{2} - \sin 0\right)\left(\tfrac{1}{5} - \tfrac{1}{7}\right) = 1\left(\tfrac{7-5}{35}\right) = \tfrac{2}{35}$$

Do in Rectangular Coords:



$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{1-(x^2+y^2)} (x^3 + xy^2)\, dz = \frac{2}{35}, \text{ by inspection}$$

§15.8 #14 from Assignments

$$\sin\phi = \frac{r}{\rho} = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

$$\rho \le 2, \quad \rho \le \csc\phi = \frac{1}{\sin\phi}$$

$$\rho = \sqrt{x^2+y^2+z^2} \le \frac{\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2}}$$

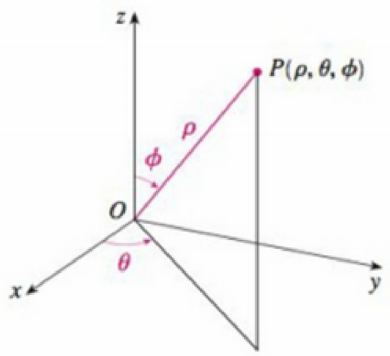
$$\frac{\rho}{\rho} = 1 \le \frac{1}{\sqrt{x^2+y^2}}$$

$$\Longrightarrow \sqrt{x^2+y^2} \le 1$$

The **spherical coordinates** (ρ, θ, ϕ) of a point P in space are shown in Figure 1, where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment OP. Note that

$$\rho \geqslant 0 \qquad 0 \leqslant \phi \leqslant \pi$$

$\sqrt{1-x^2}$ ugh!



$\Theta = \frac{3\pi}{2}$

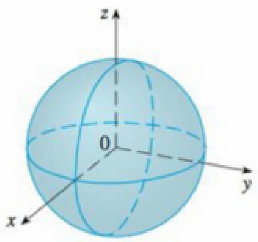


FIGURE 2 $\rho = c$, a sphere

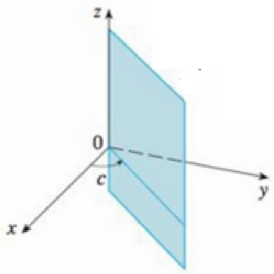


FIGURE 3 $\theta = c$, a half-plane

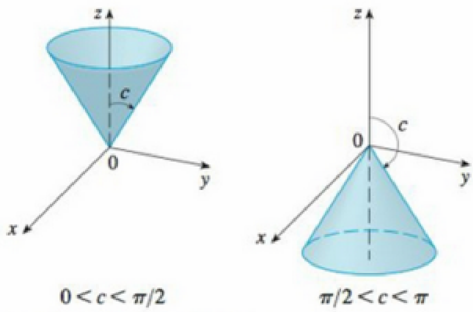


FIGURE 4 $\phi = c$, a half-cone

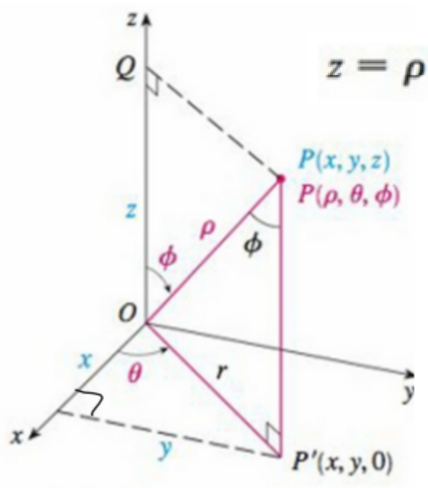


FIGURE 5

$$z = \rho \cos \phi \qquad r = \rho \sin \phi$$

$$x = \underbrace{\rho \sin \phi}_{r} \cos \theta \qquad y = \rho \sin \phi \sin \theta \qquad z = \rho \cos \phi$$

$$\rho^2 = x^2 + y^2 + z^2$$

$x = r\cos\theta$

$\frac{r}{\rho} = \sin\phi$

$r = \rho \sin\phi$

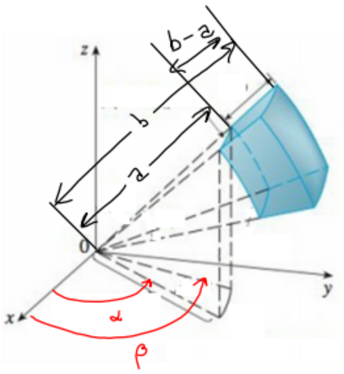
1–2 Plot the point whose spherical coordinates are given. Then find the rectangular coordinates of the point.

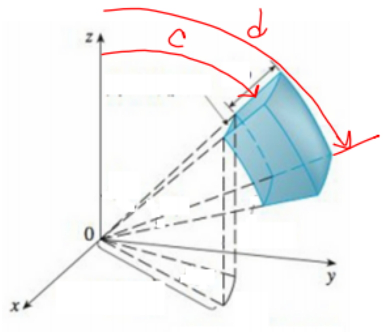
2. (a) $(5, \pi, \pi/2)$ (b) $(4, 3\pi/4, \pi/3)$

In the spherical coordinate system the counterpart of a rectangular box is a **spherical wedge** = spherical "rectangular box."

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b,\ \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d\}$$

where $a \geqslant 0$, $\beta - \alpha \leqslant 2\pi$, and $d - c \leqslant \pi$.





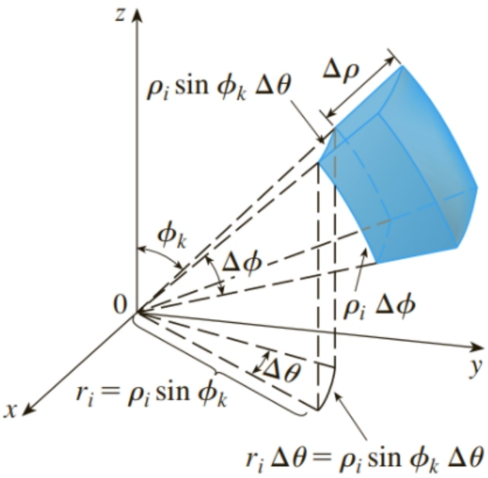


FIGURE 7

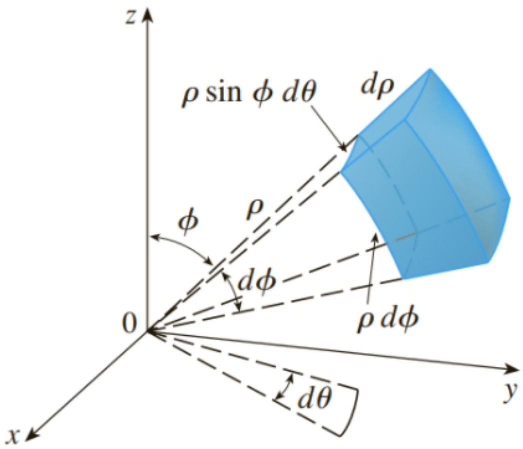


FIGURE 8

Volume element in spherical coordinates: $dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$

$$\boxed{3} \quad \iiint_E f(x, y, z) \, dV$$
$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b, \ \alpha \leqslant \theta \leqslant \beta, \ c \leqslant \phi \leqslant d\}$$