Section 14.8 Lagrange Multipliers

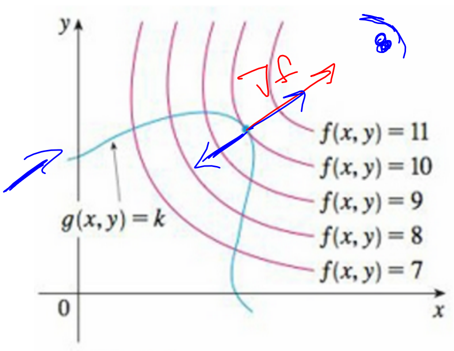


FIGURE 1

Trying to maximize or minimize f, subject to some sort of constraint g.

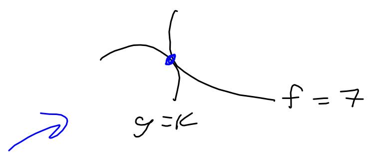
The big observation to make is that the tangent lines are the same for the level curves and the curve corresponding to $g = k$.

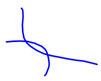
$f(x,y)$ to be optimized subject to $g(x,y) = K$

A level curve for $z = g(x,y)$

The contours of f are also level curves for $f(x,y)$

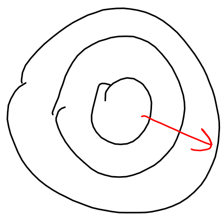
Now, the highest point on the surface $f(x,y)$ that the $g(x,y) = K$ curve hits, say (a,b) & $f(a,b)$ is highest point that $g(x,y) = K$ touches, i.e., $f(a,b) = 7$ then $g(x,y) = K$ and $f(x,y) = 7$ are tangent!





$g = K$ is a level curve.
Then $g = K$ is orthogonal to ∇g

Also $f(x,y) = 7$ is ~~also~~ also orthogonal to ∇f



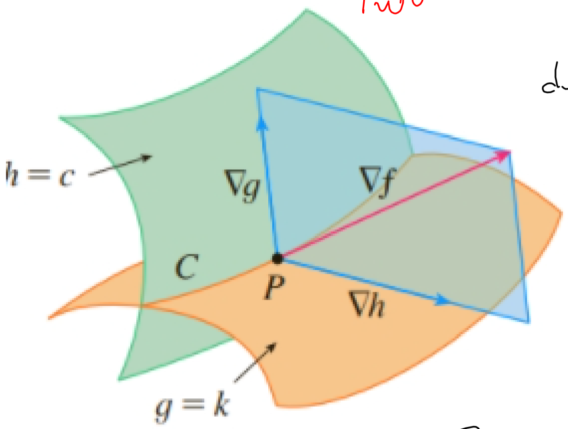
The one spot where f is maximized s.t. $g = K$, the contours $f = 7$ & $g = K$ are parallel.

So $\nabla f \perp$ contour
$\nabla g \perp$ contour

so $\nabla f = \lambda \nabla g$ at that point!
(for some $0 \neq \lambda \in \mathbb{R}$)

That's the whole idea of Lagrange Multipliers.
"fsolve" in Maple's gonna be big.

Two constraints

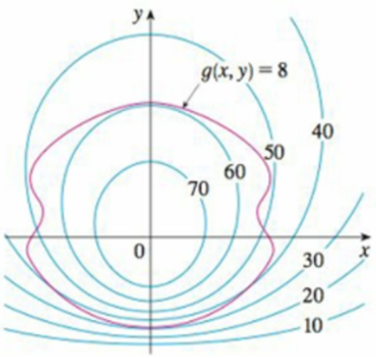


∇f is in the plane defined by ∇g & ∇f because contour for f is parallel to contour for g & h

so $\boxed{\nabla f = \lambda \nabla g + \mu \nabla h}$ for some $\lambda, \mu \in \mathbb{R}$

The set $\{\nabla g(a,b,c), \nabla h(a,b,c)\}$ spans the plane in which ∇f lies @ the optimum point.

1. Pictured are a contour map of f and a curve with equation $g(x, y) = 8$. Estimate the maximum and minimum values of f subject to the constraint that $g(x, y) = 8$. Explain your reasoning.



3–17 Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

3. $f(x, y) = x^2 + y^2$; $xy = 1$

$g(x,y) = 1$

$\nabla f = \langle 2x, 2y \rangle$ $\nabla g = \langle y, x \rangle$

$\nabla f = \lambda \nabla g$

$2x = \lambda y \Rightarrow \lambda = \frac{2x}{y}$

$2y = \lambda x \Rightarrow 2y = \frac{2x}{y} x = \frac{2x^2}{y} \Rightarrow$

$2y^2 = 2x^2 \Rightarrow$

$y = \pm x$

subject to $xy = 1$

$\Big\}$ So $y = x$

$y = \frac{1}{x}$

$y = +x$

$\Big\}$ $y = 1$, $x = 1$

3–17 Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

6. $f(x, y) = e^{xy}$; $x^3 + y^3 = 16$, $g(x,y) = x^3 + y^3$

$\nabla f = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle f(x,y) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle ye^{xy}, xe^{xy} \rangle$

$\nabla g = \langle 3x^2, 3y^2 \rangle$ s.t. $g(x,y) = 16$

$\nabla f = \lambda \nabla g \Rightarrow$

$ye^{xy} = 3\lambda x^2$ *, $xe^{xy} = 3\lambda y^2$ *

* $x=0$ forces $y=0$, b/c of $e^{xy} \neq 0$.

$\lambda = \frac{ye^{xy}}{3x^2}$ $\lambda = \frac{xe^{xy}}{3y^2}$

$\lambda = \lambda \Rightarrow \frac{ye^{xy}}{3x^2} = \frac{xe^{xy}}{3y^2}$

$\Rightarrow 3y^3e^{xy} = 3x^3e^{xy}$

$y^3 = x^3$

$y = x$ in general.

Now send that to $g(x,y) = 16$:

$x^3 + y^3 = x^3 + x^3 = 2x^3 = 16 \Rightarrow$

$x^3 = 8 \Rightarrow$

$x = 2 \Rightarrow y = 2$

$\Rightarrow f(2,2) = e^{2(2)} = e^4$

3–17 Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

10. $f(x, y, z) = x^2y^2z^2$; $x^2 + y^2 + z^2 = 1 = g(x, y)$

$\nabla f = \langle 2xy^2z^2, 2yx^2z^2, 2zx^2y^2 \rangle$

$\nabla g = \langle 2x, 2y, 2z \rangle$

$\nabla f = \lambda \nabla g \Rightarrow$

$2xy^2z^2 = 2\lambda x$, $2yx^2z^2 = 2\lambda y$, $2zx^2y^2 = 2\lambda y$

$\lambda = y^2z^2$, $\lambda = x^2z^2$, $\lambda = x^2y^2$

3 TIMES!

$y^2z^2 = x^2z^2$ $\quad y^2z^2 = x^2y^2$

$y^2 = x^2$ $\quad x^2 = z^2$

$y = \pm x$ $\quad y = \pm z$ $\quad z = \pm x$

$x^2 + y^2 + z^2 = 3x^2 = 1 \Rightarrow$

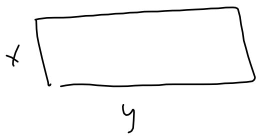
$x = \pm\sqrt{\frac{1}{3}}$

This gives $\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)$, $\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, -\frac{1}{\sqrt3}\right)$, $\left(\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)$,

$\left(\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}\right)$, $\left(-\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)$, $\left(-\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, -\frac{1}{\sqrt3}\right)$

$\left(-\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)$, $\left(-\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}\right)$

25. Use Lagrange multipliers to prove that the rectangle with maximum area that has a given perimeter p is a square.



$A = f(x,y) = xy$

s.t. $g(x,y) = 2x+2y = P$

$\nabla f = \langle y, x \rangle$

$\lambda \nabla g = \lambda \langle 2, 2 \rangle \implies y = 2\lambda, \quad x = 2\lambda$

$\longrightarrow x = y = 2\lambda$!

27–39 Use Lagrange multipliers to give an alternate solution to the indicated exercise in Section 15.7.

39. Find the shortest distance from the point $(2, 1, -1)$ to the plane $x + y - z = 1$.

Let $(x,y,z) \in P$. Then distance to $(2,1,-1)$ is

$$d = \sqrt{(x-2)^2 + (y-1)^2 + (z+1)^2} \Rightarrow$$

$f(x,y,z) = (x-2)^2 + (y-1)^2 + (z+1)^2$ to be minimized

s.t. $x + y - z = g(x,y,z) = 1$

$$\nabla f = \langle 2(x-2), 2(y-1), 2(z+1)\rangle$$
$$= \langle 2x-4, 2y-2, 2z+2\rangle$$
$$\lambda \nabla g = \lambda \langle 1, 1, -1\rangle$$

$\Rightarrow$ $\lambda = 2x-4$, $\lambda = 2y-2$, $-\lambda = 2z+2$

$2x = \lambda + 4$, $x = \frac{\lambda+4}{2}$; $2y = \lambda + 2$, $y = \frac{\lambda+2}{2}$; $2z = -\lambda - 2$, $z = \frac{-\lambda-2}{2}$

$$\frac{\lambda+4}{2} + \frac{\lambda+2}{2} - \frac{-\lambda-2}{2} = 1$$
$$\lambda + 4 + \lambda + 2 + \lambda + 2 = 2$$
$$3\lambda = -6$$
$$\boxed{\lambda = -2}$$

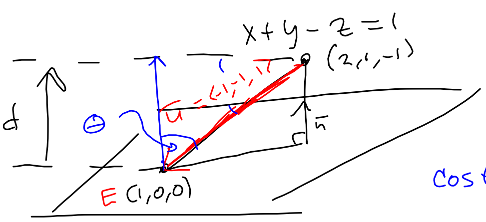
$x = \frac{-2+4}{2} = 1 = x$

$y = \frac{-2+2}{2} = 0 = y$

$z = 0$

$(1,0,0)$ is closest!

$$d = \sqrt{(x-2)^2 + (y-1)^2 + (z+1)^2}$$
$$d = \sqrt{(1-2)^2 + (0-1)^2 + (0+1)^2} = \boxed{\sqrt{3}}$$



$\bar{n} = \langle 1, 1, -1\rangle$

$\bar{u} = \langle -1, -1, 1\rangle$

$$\cos\theta = \frac{d}{\|\bar{u}\|} = \frac{|\bar{u}\cdot\bar{n}|}{\|\bar{u}\|\,\|\bar{n}\|} \Rightarrow$$

$$d = \frac{|\bar{u}\cdot\bar{n}|}{\|\bar{n}\|} = \frac{|-1-1-1|}{\sqrt{3}} = \frac{3}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{3}$$

$$\boxed{= \sqrt{3}}$$

$$\text{comp}_{\bar{n}}\,\bar{u} = \frac{\bar{u}\cdot\bar{n}}{}$$

45. (a) Find the maximum value of

$$f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$$

given that $x_1, x_2, \ldots, x_n$ are positive numbers and $x_1 + x_2 + \cdots + x_n = c$, where c is a constant.

(b) Deduce from part (a) that if $x_1, x_2, \ldots, x_n$ are positive numbers, then

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leqslant \frac{x_1 + x_2 + \cdots + x_n}{n}$$

This inequality says that the geometric mean of n numbers is no larger than the arithmetic mean of the numbers. Under what circumstances are these two means equal?