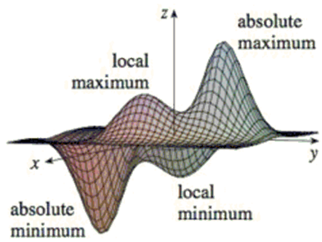


FIGURE 1

1 DEFINITION A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

The higher-dimensional version of the 1st Derivative Test.

2 THEOREM If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

EXAMPLE 1 Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$.

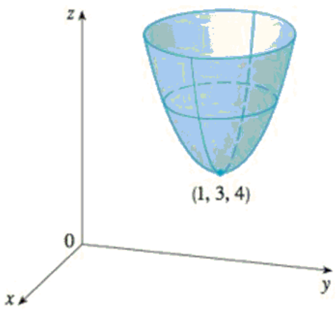


FIGURE 2
$z = x^2 + y^2 - 2x - 6y + 14$

$f_x = 2x - 2 \overset{\text{SET}}{=} 0 \Rightarrow x = 1$

$f_y = 2y - 6 \overset{\text{SET}}{=} 0 \Rightarrow y = 3$

$f(1,3) = 1^2 + 3^2 - 2(1) - 6(3) + 14$

$= 1 + 9 - 2 - 18 + 14 = 4$ ✓

Confirms the idea (NOT Proof)

Houston, we have a problem. Notice that it says IF you have an extreme AND the function is continuously differentiable (continuous derivatives of up to 2nd order, THEN the partials will be zero. Do NOT confuse this with the converse:

CONVERSE: If the partials are zero then you have an extreme.

Not generally true. Sure, we'll find where the partials are zero as CANDIDATES, but we need either a picture or some other test to seal the deal.

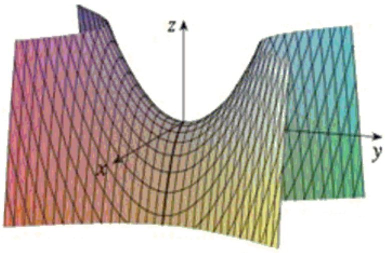


FIGURE 3
$z = y^2 - x^2$

Here's a counterexample to the converse. Partials are zero at the origin, but the origin is a saddle point. In the y-direction, it's a minimum. In the x-direction it's a maximum. In general, then, it is neither!

This motivates the following:

3 SECOND DERIVATIVES TEST Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

$\rightarrow f_{xy} f_{yx} = f_{yx} f_{yx} = (f_{yx})^2$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum. ← - - - Saddle Point

NOTE 1 In case (c) the point (a, b) is called a **saddle point** of f and the graph of f crosses its tangent plane at (a, b).

NOTE 2 If $D = 0$, the test gives no information: f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point of f.

NOTE 3 To remember the formula for D, it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

Note that Clairaut tells us $f_{xy} = f_{yx}$ if f is continuously differentiable, and virtually all of these ARE, so the mixed partials in the upper right and lower left are the same. Note also that it doesn't matter whether f_{xx} is in the top left or the bottom right. So really just have to remember that the upper left and lower right are when the 2nd partials go and the other two corners are where the ***mixed*** 2nd partials go.

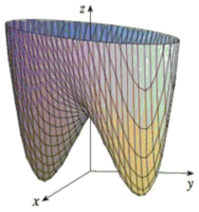


FIGURE 4
$z = x^4 + y^4 - 4xy + 1$

If you didn't know it was a pair of hip waders, you might think it's 2 mountains separated by a saddle.

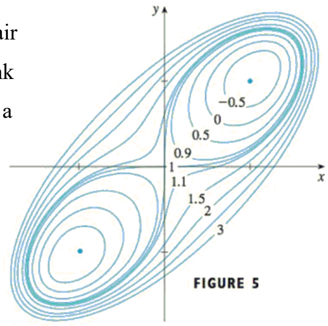


FIGURE 5

We now have ANOTHER way to minimize (find) the shortest distance from a point to a plane.

EXAMPLE 5 Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$. $\longrightarrow$ $z = 4 - x - 2y$

Distance from A to B is

$d = \sqrt{(x-1)^2 + (y-0)^2 + (z-(-2))^2}$

$(x,y,z) \in P \Longrightarrow$

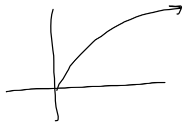
Distance $= \sqrt{(x-1)^2 + y^2 + (4-x-2y+2)^2}$

to be minimized

$f(x) = \sqrt{x}$ is an increasing function. So to minimize $f(g(x)) = \sqrt{g(x)}$, it suffices to minimize the input $g(x)$

Let $f(x,y) = \text{Distance}^2$

$= \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}^{\,2}$



$f(x,y) = (x-1)^2 + y^2 + (6-x-2y)^2$

is to be minimized

$D = 24 > 0$. Now check: $f_x = 0 \Longrightarrow x = \frac{7}{2} - y$

& $f_y = 0 \Longrightarrow x = 6 - \frac{5y}{2}$

$\frac{7}{2} - y = 6 - \frac{5}{2}y$

$7 - 2y = 12 - 5y$

$3y = 5$

$\boxed{y = \frac{5}{3}}$

$y = \frac{5}{3} \Longrightarrow x = \frac{7}{2} - \frac{5}{3} = \frac{21-10}{6} = \boxed{\frac{11}{6} = x} \longrightarrow$

$(x,y) = (\frac{11}{6}, \frac{5}{3})$

$f_{xx} = 24 > 0 \Longrightarrow$ Minimum.

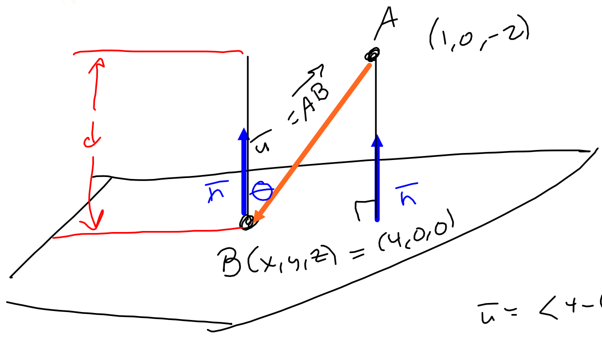
$(f_{yy} = 10 > 0)$

$d(\frac{11}{6}, \frac{5}{3}) = \boxed{\frac{5\sqrt{6}}{6} = \text{distance.}}$

$\{x = \frac{7}{2} - y, y = y\}$

$\{x = 6 - \frac{5y}{2}, y = y\}$

EXAMPLE 5 Find the shortest distance from the point (1, 0, −2) to the plane $x + 2y + z = 4$.



$y = z = 0 \Longrightarrow x = 4 \leadsto (4,0,0) = B$

$$\cos\theta = \frac{d}{\|\bar{u}\|} = \frac{\bar{u}\cdot\bar{n}}{\|\bar{u}\|\|\bar{n}\|}$$

$$d = \frac{\bar{u}\cdot\bar{n}}{\|\bar{n}\|}$$

$\bar{u} = \langle 4-1, 0, 2\rangle = \langle 3,0,2\rangle$

$\bar{n} = \langle 1,2,1\rangle \Longrightarrow \bar{u}\cdot\bar{n} = 3+0+2 = 5$

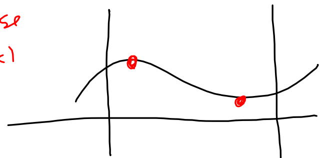
$\|\bar{n}\| = \sqrt{1^2+2^2+1^2} = \sqrt{6}$

Let $\bar{u} = \langle x-1, y, z+2\rangle$

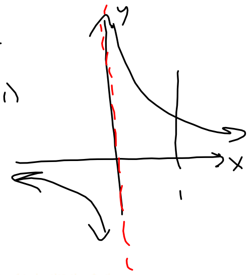
$$\Longrightarrow d = \frac{\bar{u}\cdot\bar{n}}{\|\bar{n}\|} = \frac{5}{\sqrt{6}} = \frac{5\sqrt{6}}{6}$$ ✓

Closed disk of radius 1:

2-D case
$y = f(x)$



$\frac{1}{x}$ on $(0,1)$



$$D = \{(x, y) \mid x^2 + y^2 \leq 1\}$$

$x^2+y^2=1$ is boundary

To say that a set is closed is to say that it contains its own boundary.

Higher-dimensional take on EVT:

$x=0$

8 EXTREME VALUE THEOREM FOR FUNCTIONS OF TWO VARIABLES If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

open disk: $\{(x,y) \mid x^2+y^2<1\}$
No help. Not closed set.

$\frac{1}{x}$ on $[0,1]$
but isn't continuous @ $x=0$.

Higher-dimensional take on the closed-interval method:

9 To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

The point is that extremes can occur on the boundary, and you can't guarantee that a boundary point is included in the stated domain if the domain doesn't contain its boundary.

EXAMPLE 7 Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \leqslant x \leqslant 3, 0 \leqslant y \leqslant 2\}$.

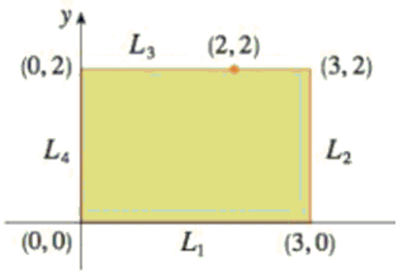


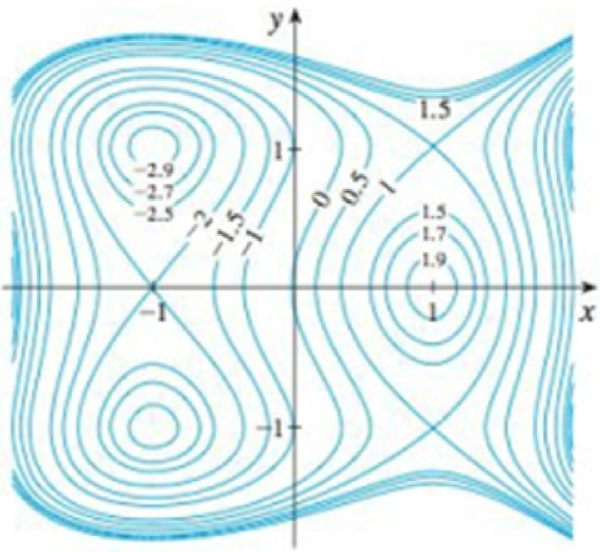
FIGURE 12

1. Suppose $(1, 1)$ is a critical point of a function f with continuous second derivatives. In each case, what can you say about f?

(a) $f_{xx}(1, 1) = 4, \quad f_{xy}(1, 1) = 1, \quad f_{yy}(1, 1) = 2$

(b) $f_{xx}(1, 1) = 4, \quad f_{xy}(1, 1) = 3, \quad f_{yy}(1, 1) = 2$

4. $f(x, y) = 3x - x^3 - 2y^2 + y^4$



5–18 Find the local maximum and minimum values and saddle point(s) of the function. If you have three-dimensional graphing software, graph the function with a domain and viewpoint that reveal all the important aspects of the function.

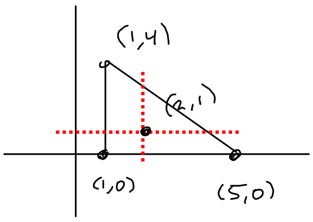
5. $f(x, y) = 9 - 2x + 4y - x^2 - 4y^2$

6. $f(x, y) = x^3y + 12x^2 - 8y$ #s 5, 6 NA

7. $f(x, y) = x^4 + y^4 - 4xy + 2$

10. $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$

30. $f(x, y) = 3 + xy - x - 2y$, D is the closed triangular region with vertices (1, 0), (5, 0), and (1, 4)



$f_x = y - 1 \overset{SET}{=} 0 \Rightarrow y = 1$

$f_y = x - 2 \overset{SET}{=} 0 \Rightarrow x = 2$

$f_{xx} = 0 = f_{yy}$, $f_{xy} = 1$

$D = f_{xx}f_{yy} - f_{xy}^2 = -1 \Rightarrow$ Neither!

Looks like a saddle point,

(1,0) to (1,4):

$x=1$ $f(1,y) = 3 + y - 1 - 2y$

$= 2 - y$ max @ $y=0$; $f(1,0) = 2$

min @ $y = 4$; $f(1,4) = -2$

(1,0) to (5,0):

$y=0$; $f(x,0) = 3 - x$ is max @ $x = 0$: $f(0,0) = 3$

min @ $x = 5$: $f(5,0) = 3 - 5 = -2 = f(5,0)$

(1,4) to (5,0)

$y = m(x - x_1) + y_1$ $y = -1(x-5) + 0$

$m = \frac{0-4}{5-1} = \frac{-4}{4} = -1$ $= -x + 5$

$f(5,0) = f(1,4) = -2$ MIN

$f(0,0) = 3$ MAX

$f(x, -x+5) = 3 + x(-x+5) - x - 2(-x+5)$

$= 3 - x^2 + 5x - x + 2x - 10$

$x \in [1,5]$ $= -x^2 + 6x - 7 = g(x) = f(x, -x+5)$

$g'(x) = -2x + 6 \overset{SET}{=} 0 \Rightarrow x = 3 \Rightarrow y = -3 + 5 = 2 \rightarrow (3,2)$

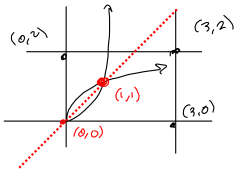
$g(3) = -3^2 + 6(3) - 7 = -9 + 18 - 7 = 2$ $f(3,2) = 2$

$g(1) = -1^2 + 6(1) - 7 = -1 + 6 - 7 = -2$ $(f(1,4) = -2$

Already did this and already checked $f(5,0)$ @ the other endpoint

30. $f(x, y) = 3 + xy - x - 2y$, D is the closed triangular region with vertices (1, 0), (5, 0), and (1, 4)

33. $f(x, y) = x^4 + y^4 - 4xy + 2$,
$D = \{(x, y) \mid 0 \leq x \leq 3, 0 \leq y \leq 2\}$



$f_x = 4x^3 - 4y \overset{SET}{=} 0 \Rightarrow y = x^3$

$f_y = 4y^3 - 4x \overset{SET}{=} 0 \Rightarrow x = y^3$ OR $y = \sqrt[3]{x}$

$f_{xx} = 12x^2$, $f_{yy} = 12y^2$

$f_{xy} = -4$, so $D = 144x^2y^2 - (-4)^2 = 144x^2y^2 - 16$

& $D(1,1) = 144 - 16 > 0$.

& $f_{xx}(1,1) = 12 > 0 \Rightarrow$ MIN. (Local)

$D(0,0) = -16 < 0 \Rightarrow$ Neither

Local min @ (1,1) of $f(1,1) = 1 + 1 - 4 + 2 = 0$

$f(1,1) = 0$ local min

$x=0$, $y=0$, $y=2$, $x=3$

$f(0,y) = y^4 + 2$ max when y is max :

$f(0,2) = 2^4 + 2 = 16 + 2 = 18 = f(0,2)$

$f(0,0) = 2$

$y=0$: $x^4 + 2 = f(x,0)$

$f(0,0)$ DONE

$f(3,0) = 3^4 + 2 = 83 = f(3,0)$

$y=2$; $x=3$

$f(x,2) = x^4 + 2^4 - 4x \cdot 2 + 2 = x^4 + 16 - 8x + 2 = x^4 - 8x + 18 = g(x)$

Maximize this on $[0,3]$

$g'(x) = 4x^3 - 8 \overset{SET}{=} 0$

$4x^3 = 8$

$x^3 = 2$

$x = \sqrt[3]{2}$

$(\sqrt[3]{2}, g(\sqrt[3]{2}))$

$g(0) = 18$

$g(3) = 81 - 8(3) + 18$

$= 81 - 24 + 18$

$= 81 - 6 = 75 = f(3,2)$

$g(\sqrt[3]{2}) = (\sqrt[3]{2})^4 - 8\sqrt[3]{2} + 18$

$= 2\sqrt[3]{2} - 8\sqrt[3]{2} + 18$

$= -6\sqrt[3]{2} + 18$

$10.44047370 \approx f(\sqrt[3]{2}, 2)$

33. $f(x, y) = x^4 + y^4 - 4xy + 2$,
$D = \{(x, y) \mid 0 \leq x \leq 3, 0 \leq y \leq 2\}$

$x=3$: $f(3,y) = 3^4 + y^4 - 4(3)(y) + 2$

$= 81 + y^4 - 12y + 2 = y^4 - 12y + 83 = h(y)$

Looks messy & time-consuming.

Do the analysis to min & max $y^4 - 12y + 83$
$y \in [0,2]$

$h'(y) = 4y^3 - 12 \overset{SET}{=} 0$

$4y^3 = 12$

$y^3 = 3$

$y = \sqrt[3]{3}$, $x = 3$

$h(\sqrt[3]{3}) = f(3, \sqrt[3]{3}) = \sqrt[3]{3}^4 - 12\sqrt[3]{3} + 83$

$= 3\sqrt[3]{3} - 12\sqrt[3]{3} + 83$

$= -9\sqrt[3]{3} + 83$

$\approx 70.01975387 \approx f(3, \sqrt[3]{3})$

$f(3,2) = 75$ is done

$f(3,0) = h(0) = 83 = f(3,0)$

$f(0,0) = 2$ MIN

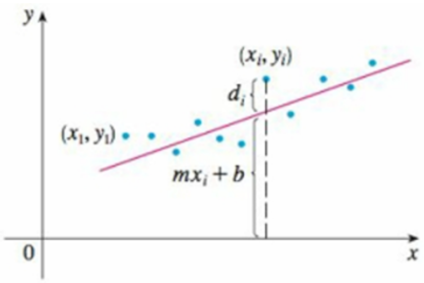
$f(3,0) = 83$ MAX

39. Find the shortest distance from the point $(2, 1, -1)$ to the plane $x + y - z = 1$.

40. Find the point on the plane $x - y + z = 4$ that is closest to the point $(1, 2, 3)$.

45. Find the maximum volume of a rectangular box that is inscribed in a sphere of radius r.

55. Suppose that a scientist has reason to believe that two quantities x and y are related linearly, that is, $y = mx + b$, at least approximately, for some values of m and b. The scientist performs an experiment and collects data in the form of points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, and then plots these points. The points don't lie exactly on a straight line, so the scientist wants to find constants m and b so that the line $y = mx + b$ "fits" the points as well as possible. (See the figure.)



Let $d_i = y_i - (mx_i + b)$ be the vertical deviation of the point (x_i, y_i) from the line. The **method of least squares** determines m and b so as to minimize $\sum_{i=1}^{n} d_i^2$, the sum of the squares of these deviations. Show that, according to this method, the line of best fit is obtained when

$$m \sum_{i=1}^{n} x_i + bn = \sum_{i=1}^{n} y_i$$

$$m \sum_{i=1}^{n} x_i^2 + b \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i$$

Thus the line is found by solving these two equations in the two unknowns m and b. (See Section 1.2 for a further discussion and applications of the method of least squares.)