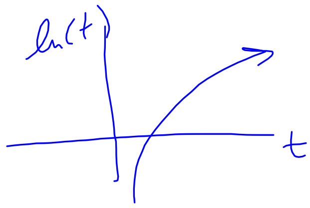
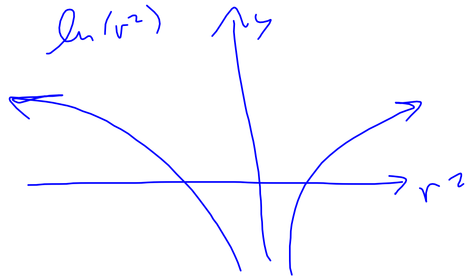
§14.2 #4

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$$

$$= \lim_{r\to 0^+} r^2 \ln(r^2) = 0 \cdot -\infty$$




$$= \lim_{r\to 0^+} \frac{r^2}{\dfrac{1}{\ln(r^2)}} =$$

$\ln(r^2) \to -\infty$

$\dfrac{1}{\ln(r^2)} \to 0^-$

$$\frac{r^2}{\dfrac{1}{\quad}} \xrightarrow[L'H]{r\to 0^+}$$

Teacher couldn't differentiate

$$\frac{1}{\ln(r^2)} = \ln(r^2)^{-1} = -1\ln(r^2)^{-2}\left(\frac{2r}{r^2}\right)$$

Keenan suggests the following:

$$r^2 \ln(r^2) = \frac{\ln(r^2)}{\dfrac{1}{r^2}} = \frac{-\infty}{\infty} \xrightarrow[L'H]{r\to 0} \frac{\dfrac{2r}{r^2}}{-2r^{-3}} = \frac{\dfrac{1}{r}}{\dfrac{-r^{-3}}{1}}$$

$\dfrac{1}{r^2} \to r^{-2}$

Thx, Keenan.

$$= \frac{-r^3}{r} = -r^2 \xrightarrow{r\to 0} 0$$

§14.5 #4

$z = \arctan\left(\frac{y}{x}\right)$ $x = e^t,\ y = 1 - e^{-t}$

Test 2 Wednesday.

For $\frac{\partial z}{\partial y}$ piece.

$\frac{y}{x} = \left(\frac{1}{x}\right)(y)$ where

$x \equiv \text{constant}$

want $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$

$$= \left(\frac{1}{1+\left(\frac{y}{x}\right)^2}\right)\left(\frac{-y}{x^2}\right)\left(e^t\right) + \left(\frac{1}{1+\left(\frac{y}{x}\right)^2}\right)\left(\frac{1}{x}\right)\left(e^{-t}\right)$$

$$\frac{d}{dx}\left[\arctan(x)\right] = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left[\arctan(u(x))\right] = \left(\frac{1}{1+(u(x))^2}\right)\left(\frac{du}{dx}\right)$$

§14.6 #s 4, 8, 9, 16, 19, 24, 34, 40, 52 → Rully tough one!

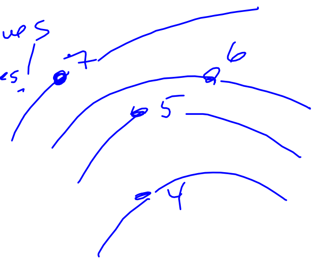
§14.7 #s 3, 6, 11, 19, 21, 26, 30, 33, 39, 40

§14.8 #s 2, 3, 6, 10, 13, 14, 27, 28, 41, 45, 46

MOAR in 14.6.

$f(x,y,z) = u$

Level curves ARE Surfaces!




Section 14.6

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$z = f(x, y)$

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

Suppose that we now wish to find the rate of change of $z$ at $(x_0, y_0)$ in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$.

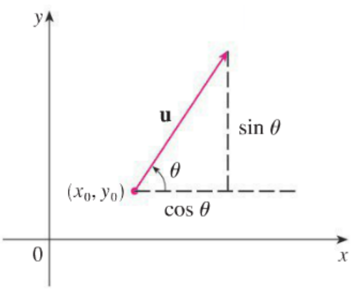



**FIGURE 2**
A unit vector $\mathbf{u} = \langle a, b \rangle = \langle \cos\theta, \sin\theta \rangle$
$\mathbf{u} = \langle a, b \rangle = \langle \cos u, \sin u \rangle$

Recall: $\text{comp}_{\bar{u}}\, \bar{v} = \dfrac{\bar{v} \cdot \bar{u}}{\|\bar{u}\|}$

If $\|\bar{u}\| = 1$, then

$\text{comp}_{\bar{u}}\, \bar{v} = \bar{v} \cdot \bar{u} = \bar{u} \cdot \bar{v}$

Not too sure what that last bit is...

14.6 Directional Derivatives and the Gradient Vector

Where we try to convince you that it *all* boils down to the gradient and that the partials with respect to $x$ and $y$ **entirely** describe the "tilt" of a surface at a point. First, we define **Directional Derivative**.

**2 Definition** The **directional derivative** of $f$ at $(x_0, y_0)$ in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

This is the "tilt" of the surface in the direction of **u**.

**3 Theorem** If $f$ is a differentiable function of $x$ and $y$, then $f$ has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

KHAN Academy

Define $g(h) = f(x_0 + ha, y_0 + hb)$ by basically holding all the other variables fixed.

$$\boxed{4} \quad g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} = D_{\mathbf{u}} f(x_0, y_0)$$

On the other hand, we can write $g(h) = f(x, y)$, where $x = x_0 + ha$, $y = y_0 + hb$, so the Chain Rule (Theorem 14.5.2) gives

$f(x,y) = f(x(h), y(h))$

$\frac{dx}{dh} = a$

$\frac{dy}{dh} = b$

See Page 978

$$\frac{df}{dh} = g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x, y)\, a + f_y(x, y)\, b$$

$x = x(h)$, $y = y(h)$,

because everything else there is fixed.

If we now put $h = 0$, then $x = x_0$, $y = y_0$, and

$$\boxed{5} \quad g'(0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b$$

$$\boxed{4} \quad g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

Comparing Equations 4 and 5, we see that

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b$$

This concludes the proof.

we're saying that $f_x \vec{i}$, $f_y \vec{j}$ spans the tangent plane, and can give you $D_{\vec{u}} f$, by writing $\vec{u} = a\vec{i} + b\vec{j}$ & $D_{\vec{u}} f = f_x$

When the angle that **u** makes with the positive $x$-axis is handy, and since **u** is of length 1, we obtain:

If the unit vector **u** makes an angle $\theta$ with the positive $x$-axis (as in Figure 2), then we can write $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ and the formula in Theorem 3 becomes

$$\boxed{6} \qquad D_{\mathbf{u}} f(x, y) = f_x(x, y) \cos\theta + f_y(x, y) \sin\theta$$

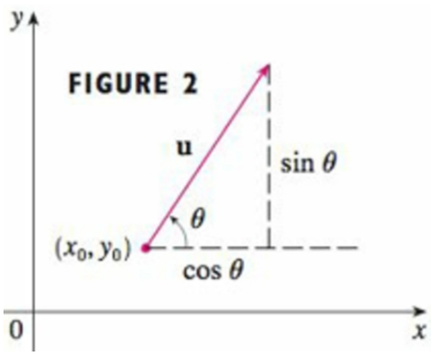


FIGURE 2

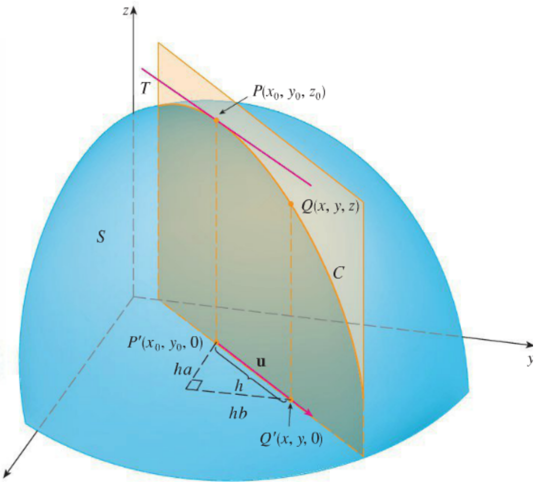


FIGURE 3

$\overrightarrow{P'Q'} = h\mathbf{u} = \langle ha, hb \rangle$ for some (nonzero) $h$.

$x - x_0 = ha$, $y - y_0 = hb$, so $x = x_0 + ha$, $y = y_0 + hb$

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

Suppose you're a monkey and you can move as fast straight up as any other direction. In a race to the top, you're going to take the path of steepest ascent. This path of steepest descent is

Gradient Vector:

Notice that you could write the directional derivative as a dot product?

$\bar{u} = \langle a, b \rangle = \langle u_1, u_2 \rangle$ , eventually $\bar{u} = \langle u_1, u_2, \dots, u_n \rangle$

$\|\bar{u}\| = 1$

$$D_{\bar{u}} f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b$$

$$= \underbrace{\langle f_x(x,y), f_y(x,y) \rangle}_{\boxed{\nabla f \text{ GRADIENT}}} \cdot \underbrace{\langle a, b \rangle}_{\bar{u}} = \nabla f \cdot \bar{u}$$

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$$

$$\text{OR } \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right\rangle$$

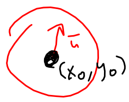
$$\nabla f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$f(x_0 + ha, y_0 + hb)$

Let $\bar{r}_0 = \langle x_0, y_0 \rangle$, $\bar{u} = \langle a, b \rangle$

Then $f(x,y) = f(\bar{r}_0 + h\bar{u})$

could be done w/ VECTOR INPUTS.



$$D_{\bar{u}} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} = \lim_{h \to 0} \frac{f(\bar{r}_0 + h\bar{u}) - f(\bar{r}_0)}{h}$$

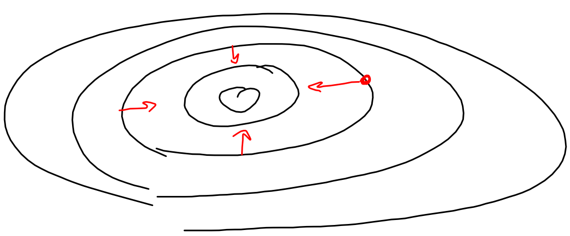
Book uses $\bar{x}_0$

$f(x)$ & $f(\bar{x})$ mean different things.

**15 THEOREM** Suppose $f$ is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

$D_{\bar{u}} f(x) = \nabla f \cdot \bar{u} = \|\nabla f\| \|\bar{u}\| \cos\theta$, where $\theta$ is the angle between $\nabla f$ & $\bar{u}$.

It's max when $\cos\theta$ is max & that means when $\cos\theta = 1 \Rightarrow \theta = 0^\circ \Rightarrow$ same direction for $\bar{u}$ & $\nabla f$.



**11–17** Find the directional derivative of the function at the given point in the direction of the vector $\mathbf{v}$.

**11.** $f(x, y) = 1 + 2x\sqrt{y}$, $(3, 4)$, $\mathbf{v} = \langle 4, -3 \rangle$

$(u_1, u_2)$ $\quad (3, 4, f(3,4))$

$\langle f_x(3,4) + 3,\ f_y(3,4) + 4,$

$u_1 - 3 = f_x(3,4)$

**20.** Find the directional derivative of $f(x, y, z) = xy + yz + zx$ at $P(1, -1, 3)$ in the direction of $Q(2, 4, 5)$.

$\bar{v} = \langle 1, 5, 2 \rangle$ isn't a unit vector.

$$\bar{u} \equiv \frac{1}{\|\bar{v}\|}\bar{v} = \frac{1}{\sqrt{1+25+4}}\langle 1,5,2\rangle = \frac{1}{\sqrt{30}}\langle 1,5,2\rangle$$

$$\nabla f \circ \bar{u} = \langle f_x, f_y, f_z \rangle \circ \left(\frac{1}{\sqrt{30}}\langle 1,5,2\rangle\right)$$

$$= \frac{1}{\sqrt{30}}\langle f_x, f_y, f_z \rangle \circ \langle 1, 5, 2 \rangle$$

**20.** Find the directional derivative of $f(x, y, z) = xy + yz + zx$ at $P(1, -1, 3)$ in the direction of $Q(2, 4, 5)$.

$f_x = y + z$, $f_y = x + z$, $f_z = y + x$

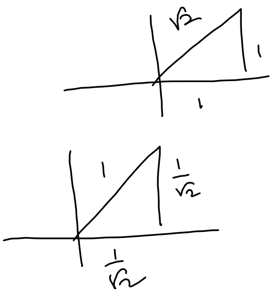
$$\nabla f \circ \bar{u} = \frac{1}{\sqrt{30}}\langle y+z, x+z, x+y \rangle \circ \langle 1, 5, 2 \rangle$$

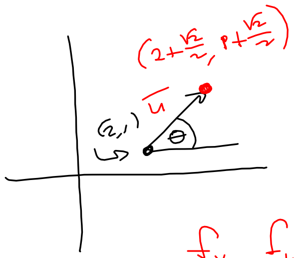
$$= \frac{1}{\sqrt{30}}\left[y+z + 5(x+z) + 2(x+y)\right] = \frac{1}{\sqrt{30}}\left[7x + 3y + 6z\right]$$

$$= D_{\bar{u}}\left(f(x,y,z)\right)$$

4. $f(x, y) = x^2y^3 - y^4$, $(2, 1)$, $\theta = \pi/4$

$(2,1)$




$$\vec{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\sqrt{2}}\langle 1, 1\rangle = \frac{\sqrt{2}}{2}\langle 1, 1\rangle$$

$f_x$, $f_y$ & $\langle f_x, f_y\rangle \cdot \vec{u}$