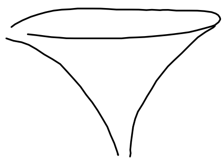
§14.1 #35

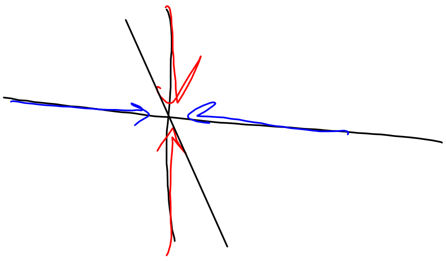


$$\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2(y)}{2x^2+2y^2}$$

y-axis: $\frac{0}{0+2} = 0$

x-axis $\frac{x^2\cdot 0}{2x^2} = 0$

$y = mx$ $\quad \frac{x^2\sin^2(mx)}{2x^2+2(mx)^2} = \frac{x^2\sin^2(mx)}{(2+2m^2)x^2} = \frac{\sin^2(mx)}{2+2m^2} \xrightarrow{(x,y)\to(0,0)} \frac{0}{2+2m^2} = 0$



$$\lim_{(x,y)\to(3,7)}$$

or along $x=3$

$y = m(x-3)+7$

or along $y=7$

Section 14.3 Partial Derivatives

~~§14.3 #s 4, 10, 11, 13, 15, 18, 21, 26, 29, 30, 47, 50, 52, 53, 56, 59, 71*, 78, 81, 83~~

~~*Clairaut.~~

$$f_x(a,b) = f_x(a, b) = g'(a) \quad \text{where} \quad g(x) = f(x, b)$$

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

$$f_y(a,b) = f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}$$

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a,b)}{h}$$

$\boxed{4}$ If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

$f_x = \frac{\partial f}{\partial x}$

" ∂ " = "del"

$f(x,y) = x^2y^2$

$\Rightarrow f_x = 2xy^2$

$f_y = 2x^2y$

$f_{xy} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = 4xy$

$f_{yx} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = 4xy$

Mixed 2^{nd} Partials Agree!

Clairaut says if it's smooth, then

$f_{xy} = f_{yx}$

Notations for Partial Derivatives If $z = f(x, y)$, we write

∂ = "del"

$z = f(x, y)$

$$f_x = f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y = f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

$$f(x,y) = x^2 \sin(xy) + x^4 y^5$$ Find $f_x(x,y)$, $f_y(x,y)$

$f_x = 2x \sin(xy) + x^2 \cos(xy) \cdot y + 3x^3 y^5$

$f_y = x^2 \cos(xy) \cdot x + 5x^4 y^4$

$\frac{\partial}{\partial x}[xy] = y$

$\frac{d}{dx}[7x] = 7$

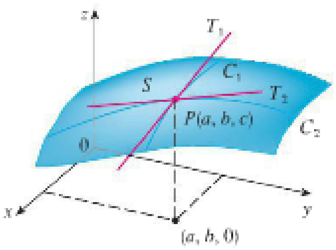


FIGURE 1
The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.

EXAMPLE 2 If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

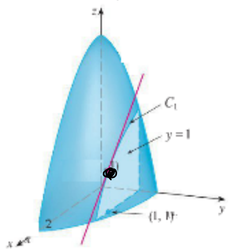
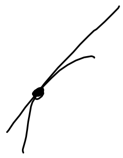


FIGURE 2



f_x vertical plane $\parallel$ to xz-plane

f_y vertical plane $\parallel$ to yz-plane.

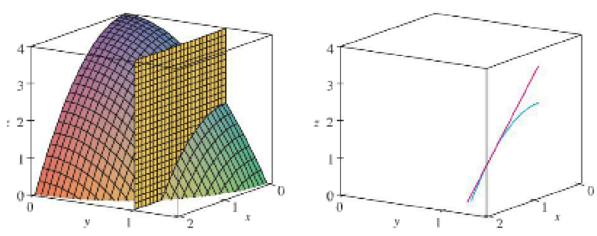


FIGURE 4 (a) (b)

Find all 1st & 2nd partials

$f\,(x^2y^3)(\sin(xy))\,g \qquad (fg)' = f'g + fg'$

$f_x = \underline{2xy^3\sin(xy)} + x^2y^3\cos(xy)\cdot y \quad = 2xy^3\sin(xy) + x^2y^4\cos(xy)$

$f_{xx} = 2y^3\sin(xy) + 2xy^3\cos(xy)\cdot y + 2xy^4\cos(xy) - x^2y^4\sin(xy)\cdot y$

$\sin(xy)(2y^3 - x^2y^5) + \cos(xy)(4xy^4)$ etc.

$f_{xy} = 6xy^2\sin(xy) + 2xy^3\cos(xy)\cdot x$

4. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in the following table.
You needn't reproduce the table if you attach the exercises as a cover sheet.

Duration (hours)

Wind speed (knots): v \ t	5	10	15	20	30	40	50
10	2	2	2	2	2	2	2
15	4	4	5	5	5	5	5
20	5	7	8	8	9	9	9
30	9	13	16	17	18	19	19
40	14	21	25	28	31	33	33
50	19	29	36	40	45	48	50
60	24	37	47	54	62	67	69

$h(v,t)$

$h(40,10) - h(40,15)$ NO

$h_v(40,15)$

$$\frac{h(30,15) - h(40,15)}{30-40} = \frac{16-25}{-10} = \frac{-9}{-10} = \frac{9}{10}$$

$$\frac{h(50,15) - h(40,15)}{50-40} = \frac{36-25}{10} = \frac{11}{10}$$

AvG: $\frac{1}{2}\left[\frac{9+11}{10}\right] = \frac{20}{20} = 1$

(a) What are the meanings of the partial derivatives $\partial h/\partial v$ and $\partial h/\partial t$?

(b) Estimate the values of $f_v(40, 15)$ and $f_t(40, 15)$. What are the practical interpretations of these values?

(c) What appears to be the value of the following limit?

$$\lim_{t\to\infty} \frac{\partial h}{\partial t}$$

$=$

$h_t(40,15)$:

$$\boxed{\frac{h(40,10) - h(40,15)}{10-15} \quad \text{Average of these} \quad \& \quad \frac{h(40,20) - h(4,15)}{20-15}}$$

Functions of 3 or more variables...

$f(x_1, x_2, \dots, x_n)$

f_{x_3}, etc. Same deal.

$1, t, t^2, t^3, t^4, t^5, \dots$

Higher Derivatives

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

f_{xy} means $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

$\frac{\partial^2 f}{\partial y \partial x}$ means the same thing.

(order is reversed from f_{xy} notation

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2}{\partial y \partial x}\left[f\right] = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$$

$$f(x,y) = x^2\sin(xy) + x^4y^5$$

Find $f_x(x,y), f_y(x,y), f_{xx}(x,y), f_{yy}(x,y), f_{xy}(x,y), f_{yx}(x,y)$

$f_x = 2x\sin(xy) + x^2(\cos(xy))\cdot y + 4x^3y^5 = 2x\sin(xy) + \boxed{x^2y\cos(xy)} + 4x^3y^5$

$f_{xx} = 2\sin(xy) + 2x(-\cos(xy))\cdot y + 12x^2y^5$

$f_{xy} = \frac{\partial^2}{\partial y\partial x}[f] = 2x(\cos(xy))\cdot x + 20x^3y^4 = 2\sin(xy) + 2x(-\cos(xy))y + 2xy\cos(xy) + x^2y(-\sin(xy))\cdot y$

$f_y = x^2(\cos(xy))\cdot x + 5x^4y^4 = x^3\cos(xy) + 5x^4y^4$

$f_{yy} =$

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

"Smoothness" means $f_{xx}, f_{xy}, f_{yy}, f_{yx}$ exist

This is stronger hypothesis

In practice, little difference, b/c most functions we see have continuous 2nd derivatives

Mean Value Theorem for Derivatives.

f is differentiable on $[a,b]$

then $\exists\, c \in (a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

It doesn't assume that f' is cnts

It's enough that it exists for MVT to hold.

Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ **harmonic functions;**

heat conduction, fluid flow, and electric potential.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

4. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in the following table.

Duration (hours)

Wind speed (knots) v \ t	5	10	15	20	30	40	50
10	2	2	2	2	2	2	2
15	4	4	5	5	5	5	5
20	5	7	8	8	9	9	9
30	9	13	16	17	18	19	19
40	14	21	25	28	31	33	33
50	19	29	36	40	45	48	50
60	24	37	47	54	62	67	69

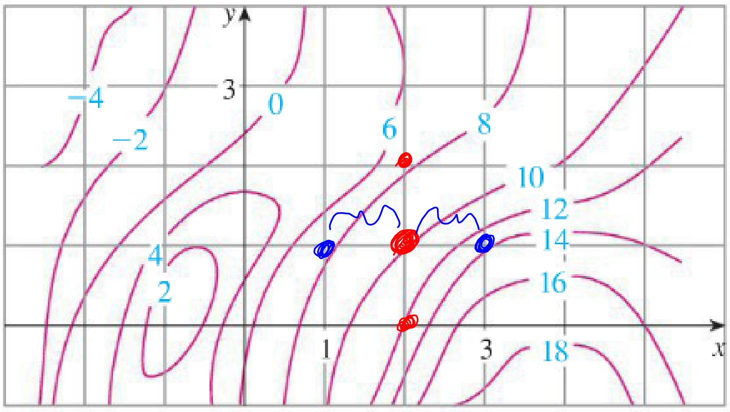
(a) What are the meanings of the partial derivatives $\partial h/\partial v$ and $\partial h/\partial t$?

(b) Estimate the values of $f_v(40, 15)$ and $f_t(40, 15)$. What are the practical interpretations of these values?

(c) What appears to be the value of the following limit?

$$\lim_{t \to \infty} \frac{\partial h}{\partial t}$$

10. A contour map is given for a function f. Use it to estimate $f_x(2, 1)$ and $f_y(2, 1)$.



11. If $f(x, y) = 16 - 4x^2 - y^2$, find $f_x(1, 2)$ and $f_y(1, 2)$ and interpret these numbers as slopes. Illustrate with either hand-drawn sketches or computer plots.

13–14 Find f_x and f_y and graph f, f_x, and f_y with domains and viewpoints that enable you to see the relationships between them.

13. $f(x, y) = x^2 + y^2 + x^2 y$ **14.** $f(x, y) = xe^{-x^2-y^2}$ #14 NA

15–40 Find the first partial derivatives of the function.

15. $f(x, y) = x^4 + 5xy^3$

18. $f(x, t) = \sqrt{3x + 4t}$

21. $f(x, y) = \dfrac{x}{y}$

26. $u(r, \theta) = \sin(r\cos\theta)$

$u_r = \cos(r\cos\theta) \cdot \cos\theta$

$u_\theta = \cos(r\cos\theta) \cdot (-r\sin\theta)$

29. $F(x, y) = \int_y^x \cos(e^t)\, dt$

30. $F(\alpha, \beta) = \int_\alpha^\beta \sqrt{t^3 + 1}\, dt$

47–50 Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$. $y = \text{constant}$ in $\frac{\partial z}{\partial x}$ stuff

47. $x^2 + 2y^2 + 3z^2 = 1$

$\frac{\partial}{\partial x}$:

$2x + 6zz' = 0$

$6zz' = -2x$

$z' = \frac{\partial z}{\partial x} = \frac{-2x}{6z}$

$\frac{\partial}{\partial y}$:

$4y + 6zz' = 0$

$6zz' = -4y$

$\frac{\partial z}{\partial y} = z' = -\frac{2y}{3z}$

$z = z(x)$ or $z = z(y)$ & chain rule for $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$, respectively.

50. $yz + x \ln y = z^2$

$\frac{\partial z}{\partial x}$: $y\frac{\partial z}{\partial x} + \ln y = 2z\frac{\partial z}{\partial x}$

$(y - 2z)\frac{\partial z}{\partial x} = -\ln y$

$$\boxed{\frac{\partial z}{\partial x} = \frac{\ln y}{2z - y}}$$

$\frac{\partial z}{\partial y}$: $\frac{\partial z}{\partial y} + \frac{x}{y} = 2z\frac{\partial z}{\partial y}$

$(1 - 2z)\frac{\partial z}{\partial y} = -\frac{x}{y}$

$$\boxed{\frac{\partial z}{\partial y} = \frac{x}{y(2z-1)}}$$

51–52 Find $\partial z/\partial x$ and $\partial z/\partial y$.

52. (a) $z = f(x)g(y)$ (b) $z = f(xy)$

(c) $z = f(x/y)$

CALC I Implicit Diff:

$$x^2y + 2x\sin y = x^2$$

Find $\frac{dy}{dx}$: $2xy + x^2y' + 2\sin y + 2x(\cos y)y' = 2x$

$$x^2y' + 2x\cos(y)y' = 2x - 2xy - 2\sin(y)$$

$$y'(x^2 + 2x\cos(y)) = 2x - 2xy - 2\sin(y)$$

$$y' = \frac{2x - 2xy - 2\sin(y)}{x^2 + 2x\cos(y)}$$

53–58 Find all the second partial derivatives.

53. $f(x, y) = x^4y - 2x^3y^2$

56. $T = e^{-2r}\cos\theta$

$T_r = -2e^{-2r}\cos\theta$

$T_{rr} = 4e^{-2r}\cos\theta$

$T_{r\theta} = 2e^{-2r}\sin\theta$

$T_\theta = -e^{-2r}\sin\theta$

$T_{\theta\theta} = -e^{-2r}\cos\theta$

$T_{\theta r} = 2e^{-2r}\sin\theta$

59–62 Verify that the conclusion of Clairaut's Theorem holds, that is, $u_{xy} = u_{yx}$.

59. $u = x^4y^3 - y^4$

$u_x = 4x^3y^3$ $u_y = 3x^4y^2 - 4y^3$

$u_{xy} = 12x^3y^2 = u_{yx} = 12x^3y^2$ Same!

71. If $f(x, y, z) = xy^2z^3 + \arcsin(x\sqrt{z})$, find f_{xzy}. $= f_{yxz}$
[*Hint:* Which order of differentiation is easiest?]

$f_y = 2xyz^3 \Rightarrow f_{yx} = 2yz^3 \Rightarrow \boxed{f_{yxz} = 6yz^2}$

81. The *diffusion equation*

$$\frac{\partial c}{\partial t} = D\,\frac{\partial^2 c}{\partial x^2}$$

where D is a positive constant, describes the diffusion of heat through a solid, or the concentration of a pollutant at time t at a distance x from the source of the pollution, or the invasion of alien species into a new habitat. Verify that the function

$$c(x, t) = \frac{1}{\sqrt{4\pi D t}}\, e^{-x^2/(4Dt)}$$

is a solution of the diffusion equation.

83. The total resistance R produced by three conductors with resistances R_1, R_2, R_3 connected in a parallel electrical circuit is given by the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Find $\partial R/\partial R_1$.

$$R^{-1} = R_1^{-1} + R_2^{-1} + R_3^{-1}$$

$$-1(R^{-2})\left(\frac{\partial R}{\partial R_1}\right) = -R_1^{-2}$$

$$-\frac{1}{R^2}\left(\frac{\partial R}{\partial R_1}\right) = -\frac{1}{R_1^2}$$

$$\frac{\partial R}{\partial R_1} = \frac{1}{R_1^2}\cdot R^2 = \frac{1}{R_1^2}\left(\frac{1}{R}\right)^{-2} = \boxed{\frac{1}{R_1^2}\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)^{-2} = \frac{\partial R}{\partial R_1}}$$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

$$R = \frac{1}{R_1^{-1} + R_2^{-1} + R_3^{-1}} \Rightarrow \frac{\partial R}{\partial R_1} = -1\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)^{-2}\left(-\frac{1}{R_1^2}\right)$$