DIVERGENCE THEOREM. THE PINNACLE OF CALCULUS III

Recall Stokes' Thm:

$$\int_C \bar{F} \cdot d\bar{r} = \iint_{S} \text{curl}\,\bar{F} \cdot d\bar{S}$$

It's a generalization of Green's Theorem, and both Green's and Stokes' call on the Fundamental Theorem for Line Integrals, which in turn calls on FTC II. It all goes back to FTC II. Continuity of the integrand guarantees differentiability of the integral, and the integral is the inverse of the differentiation operation.

FTC II says the definite integral depends only on evaluation of the antiderivative on the boundary (with a nifty subtraction, which gives it its second synonym "Net Change Theorem."). It's just really cool that Stokes comes along and generalizes everything. The boundary of a surface is a space curve, in much the same way that the boundary of an interval in FTC II consists of its endpoints.

STOKES' THEOREM Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S}$$

The line integral is telling us how "parallel" to the boundary **F** is and whether it opposes the orientation of the curve C or is in the same direction.

curl **F** is the tendency of **F** to twist. We measure how much that twisting is perpendicular to the surface.

In 16.5, we also talked about the component of **F** that's normal to the surface:

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \text{div}\,\mathbf{F}(x, y)\, dA$$

The line integral is telling us how "perpendicular" to the boundary **F** is and whether it's "in" or "out" of the surface (+ is out, - is in).

$\nabla \bullet \mathbf{F}$

div **F** is the tendency of **F** to spread or "diverge."

This is, in fact, another way of expressing Green's Theorem.

This generalizes to 3 dimensions as follows:

THE DIVERGENCE THEOREM, IN ALL ITS GLORY:

(2-D to 3-D)

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \text{div}\,\mathbf{F}(x, y, z)\, dV$$

1-D to 2-D

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \text{div}\,\mathbf{F}(x, y)\, dA$$

Actually, here's its full glory:

THE DIVERGENCE THEOREM Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div}\,\mathbf{F}\, dV$$

We won't be too worried in this, our first look, about confirming the hypotheses. I wouldn't worry too much about the definition of "simple solid region." In higher analysis, you'll spend a lot of time talking about "convex, simply-connected" regions, and the like, and you'll build up more general regions as unions of convex sets, when you try to prove more general results in Advanced Calculus.

We're generally going to be OK, always and everywhere, and the main thing to worry about is the domain of the vector field, which is usually the best indicator of where **F** might NOT have continuous partial derivatives. Look for square roots and divisions by zero, mainly. Those can trip you up, if they fall within the region E, which is the main thing that can make these theorems fail you.

PROOF Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$. Then

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

so

$$\iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E \frac{\partial P}{\partial x}\, dV + \iiint_E \frac{\partial Q}{\partial y}\, dV + \iiint_E \frac{\partial R}{\partial z}\, dV$$

is the right-hand side of the equation in the theorem. And the right-hand side of the equation is:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot \mathbf{n}\, dS$$

$$= \iint_S P\,\mathbf{i} \cdot \mathbf{n}\, dS + \iint_S Q\,\mathbf{j} \cdot \mathbf{n}\, dS + \iint_S R\,\mathbf{k} \cdot \mathbf{n}\, dS$$

Then the proof is complete if we can prove the following 3 equations, of which we will prove the 3rd, and hand-wave the rest as being essentially the same argument, re-labeled.

$$\iint_S P\,\mathbf{i} \cdot \mathbf{n}\, dS = \iiint_E \frac{\partial P}{\partial x}\, dV$$

$$\iint_S Q\,\mathbf{j} \cdot \mathbf{n}\, dS = \iiint_E \frac{\partial Q}{\partial y}\, dV$$

$$\iint_S R\,\mathbf{k} \cdot \mathbf{n}\, dS = \iiint_E \frac{\partial R}{\partial z}\, dV$$

We're assuming E is Type 1 to make this argument. The more general proof is for higher math courses, but the main idea is covered nicely, here:

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

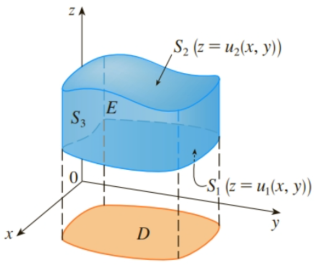


FIGURE 1

$$\iint_{S_3} R\,\mathbf{k} \cdot \mathbf{n}\, dS = \iint_{S_3} 0\, dS = 0$$

so

$$\iint_S R\,\mathbf{k} \cdot \mathbf{n}\, dS = \iint_{S_1} R\,\mathbf{k} \cdot \mathbf{n}\, dS + \iint_{S_2} R\,\mathbf{k} \cdot \mathbf{n}\, dS$$

Now, by 16.7: **9** $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$

In the case of a surface S given by a graph $\boxed{z = g(x, y)}$, we can think of x and y as parameters and use Equation 3 to write

Recall:

Also 16.7: $\displaystyle\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot \left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}\right)$

3 $\displaystyle\mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}$

The equation of S_2 is $z = u_2(x, y)$, $(x, y) \in D$, and the outward normal $\mathbf{n}$ points upward, so from Equation 16.7.10 (with $\mathbf{F}$ replaced by $R\,\mathbf{k}$) we have

$$\iint_{S_2} R\,\mathbf{k} \cdot \mathbf{n}\, dS = \iint_D R(x, y, u_2(x, y))\, dA$$

Likewise, along the bottom, where the surface points downward:

$$\iint_{S_1} R\,\mathbf{k}\cdot\mathbf{n}\,dS = -\iint_D R(x, y, u_1(x, y))\,dA$$

This gives:

$$\iint_S R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iint_D \left[R(x, y, u_2(x, y)) - R(x, y, u_1(x, y))\right] dA$$

and so,

$$\iint_S R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iiint_E \frac{\partial R}{\partial z}\,dV$$

EXAMPLE 1 Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

$$\iint_S \bar{F}\cdot d\bar{S} = \iint_S \bar{F}\cdot\bar{n}\,dS = \iiint_E \operatorname{div}\bar{F}\,dV$$

$$r(\phi,\theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$$

$$r_\phi = \langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi\rangle \quad \cos\phi\cos\theta, \cos\phi\sin\theta$$

$$\times\ r_\theta = \langle -\sin\phi\sin\theta, \sin\phi\cos\theta, 0\rangle \quad -\sin\phi\sin\theta, \sin\phi\cos\theta$$

$$\langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi\cos^2\theta + \sin\phi\cos\phi\sin^2\theta\rangle$$

$$= \langle \sin\phi\sin\theta, \sin\phi\cos\theta, \sin\phi\cos\phi\rangle = \bar{r}_\phi\times\bar{r}_\theta$$

$$\mathbf{n}\,dS = \frac{\bar{r}_\phi\times\bar{r}_\theta}{\|\bar{r}_\phi\times\bar{r}_\theta\|}\|\bar{r}_\phi\times\bar{r}_\theta\|\,d\phi\,d\theta = (\bar{r}_\phi\times\bar{r}_\theta)\,d\phi\,d\theta \qquad \mathbf{n} = \frac{\bar{r}_\phi\times\bar{r}_\theta}{\|\bar{r}_\phi\times\bar{r}_\theta\|}$$

EXAMPLE 1 Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

Tried the mouse as a pen and to do the *surface* integral of the normal component of **F** to the surface *S*.

$$\bar{F} = \langle \cos\phi, \sin\phi\sin\theta, \sin\phi\cos\theta\rangle$$

$$\iint_S \bar{F}\cdot d\bar{S} = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \langle \cos\phi, \sin\phi\sin\theta, \sin\phi\cos\theta\rangle \langle \sin\phi\sin\theta, \sin\phi\cos\theta, \sin\phi\cos\phi\rangle\,d\phi\,d\theta$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}} (\cos\phi\sin\phi\sin\theta + \sin^2\phi\sin\theta\cos\theta + \sin^2\phi\cos\phi\cos\theta)\,d\phi\,d\theta$$

$$\frac{1}{2} - \frac{1}{2}\cos(2\phi).$$

$$= \int_0^{2\pi}\left[\frac{\sin^2\phi}{2}\sin\theta + \frac{1}{2}\sin\theta\cos\theta - \frac{1}{4}\sin(2\phi)\sin\theta\cos\theta + \frac{1}{3}\sin^3\phi\cos\theta\right]_0^{\frac{\pi}{2}} d\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{2}\sin\theta + \frac{1}{4}\sin^2\theta - 0 + \frac{1}{3}\cos\theta\right)d\theta$$

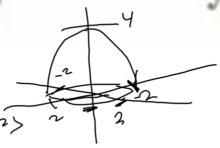
$$= \left[-\frac{1}{2}\cos\theta + \frac{1}{3}\sin\theta\right]_0^{2\pi}$$

Meh. Too slow and clumsy. Too many errors, also.

1–4 Verify that the Divergence Theorem is true for the vector field $\mathbf{F}$ on the region E.

2. $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + xy\,\mathbf{j} + z\,\mathbf{k}$,
E is the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_E \operatorname{div}\mathbf{F}(x, y, z)\,dV$$



$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 4-r^2\rangle$

$\vec{r}_r = \langle \cos\theta, \sin\theta, -2r\rangle$ $\quad \cos\theta, \sin\theta$

$\times\ \vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$ $\quad -r\sin^2\theta, r\cos\theta$

$\vec{r}_r \times \vec{r}_\theta = \langle 2r^2\cos\theta, 2r^2\sin\theta, r\cos^2\theta + r\sin^2\theta\rangle$ $\qquad \langle x^2, xy, z\rangle$

$\vec{F}(\vec{r}(r,\theta)) = \langle r^2\cos^2\theta, r^2\cos\theta\sin\theta, 4-r^2\rangle$

$\bullet\ \langle 2r^2\cos\theta, 2r^2\sin\theta, r\rangle$

$= 2r^4\cos^3\theta + 2r^4\sin^2\theta\cos\theta + 4r - r^3 = \vec{F}\bullet\vec{r}_r\times\vec{r}_\theta$

$\iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^2 \left(2r^4\cos\theta\, \boxed{-2r^4\sin^2\theta\cos\theta + 2r^4\sin^2\theta\cos\theta} + 4r - r^3\right) dr\,d\theta$

$\cos^3\theta = (1-\sin^2\theta)\cos\theta = \cos\theta - \sin^2\theta\cos\theta$

$= \int_0^2 \left[2r^4\sin\theta + 4r\theta - r^3\theta\right]_0^{2\pi} dr = \int_0^2 (4r\cdot 2\pi - r^3\cdot 2\pi)\,dr$

$= \left[8\pi\cdot\frac{r^2}{2} - \frac{2\pi}{4}r^4\right]_0^2 = 4\pi\cdot 2^2 - \frac{\pi}{2}\cdot 2^4 = 16\pi - 8\pi = \boxed{8\pi}$

Now for $\iiint$

$0 \le r \le 2,\quad 0 \le \theta \le 2\pi,\quad 0 \le z \le 4 - r^2$

$\iiint (3r\cos\theta + 1)\,dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} (3r\cos\theta + 1)\,r\,dz\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^2\int_0^{4-r^2} (3r^2\cos\theta + r)\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \left[(3r^2\cos\theta + r)z\right]_0^{4-r^2} dr\,d\theta$

$= \int_0^{2\pi}\int_0^2 (3r^2\cos\theta + r)(4 - r^2)\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^2 \left[12r^2\cos\theta - 3r^4\cos\theta + 4r - r^3\right] dr\,d\theta$

$= \int_0^{2\pi} \left[4r^3\cos\theta - \frac{3r^5}{5}\cos\theta + 2r^2 - \frac{1}{4}r^4\right]_0^2 d\theta$

$= \int_0^{2\pi} \left(32\cos\theta - \frac{96}{5}\cos\theta + 4\right) d\theta = \left(32 - \frac{96}{5}\right)\sin\theta\Big]_0^{2\pi} + 4\theta\Big]_0^{2\pi}$

$= \boxed{8\pi}$

$$\int_0^{2\cdot \mathrm{Pi}}\int_0^2\int_0^{4-r^2} (3\cdot r\cdot\cos(q) + 1)\cdot r\,dz\,dr\,dq = 8\pi$$

5–15 Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$; that is, calculate the flux of $\mathbf{F}$ across S.

5. $\mathbf{F}(x, y, z) = e^x \sin y\,\mathbf{i} + e^x \cos y\,\mathbf{j} + yz^2\,\mathbf{k}$,
S is the surface of the box bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 2$

$$\iint_{S = \partial D} \bar{F} \cdot d\bar{S} = \iiint_D \operatorname{div} \bar{F}\, dV = \int_0^1 \int_0^1 \int_0^2 2yz\, dz\, dy\, dx$$

$$\operatorname{div} \bar{F} = e^x \sin y - e^x \sin y + 2yz = 2yz \qquad = 2 \int_0^1 dx \int_0^1 y\, dy \int_0^2 z\, dz$$

$$= \left[2x\right]_0^1 \left[\tfrac{1}{2} y^2\right]_0^1 \left[\tfrac{1}{2} z^2\right]_0^2$$

$$= [2] \left[\tfrac{1}{2}\right] [2] = 2$$

i.e. Flux through the surface is $\iint_S \bar{F} \cdot d\bar{S} = 2$

8. $\mathbf{F}(x, y, z) = x^3y\,\mathbf{i} - x^2y^2\,\mathbf{j} - x^2yz\,\mathbf{k}$,
S is the surface of the solid bounded by the hyperboloid $x^2 + y^2 - z^2 = 1$ and the planes $z = -2$ and $z = 2$

1st thing: $\operatorname{div}\bar{F} = 3x^2y - 2x^2y - x^2y = 0 \Rightarrow$

$\mathcal{D}$ = solid contained in S

$\iiint_{\mathcal{D}} \operatorname{div}\bar{F}\,dV = \iiint_{\mathcal{D}} 0\,dv = 0\,!$

dV is volume increment in the parameter domain chosen for the integral
No change of variables $\Rightarrow dV = dy\,dz\,dx$
Otherwise, $dV = \|\bar{r}_u \times \bar{r}_v\|\,dz\,du\,dv$ if 2-parameter sub. (cylindrical)

3-parameter $|\bar{r}_u \cdot (\bar{r}_v \times \bar{r}_w)|$

polar & cylindrical:
$\|\bar{r}_r \times \bar{r}_\theta\| = r$
Spherical:
$\rho^2 \sin\phi = |\bar{r}_\rho \cdot (\bar{r}_\phi \times \bar{r}_\theta)|\,d\rho\,d\phi\,d\theta$

9. $\mathbf{F}(x, y, z) = xy \sin z\, \mathbf{i} + \cos(xz)\, \mathbf{j} + y \cos z\, \mathbf{k}$,
S is the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$

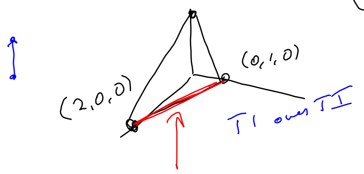
Do-able, but a little abstract for us, right now.

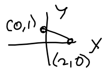
10. $\mathbf{F}(x, y, z) = x^2 y\, \mathbf{i} + xy^2\, \mathbf{j} + 2xyz\, \mathbf{k}$,
S is the surface of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + 2y + z = 2$

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_D \operatorname{div} \vec{F}\, dV$$

$(2,0,0), (0,1,0), (0,0,2)$

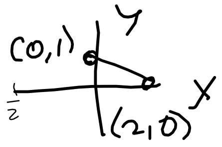
$$\operatorname{div} \vec{F} = 2xy + 2xy + 2xy = \boxed{6xy = \operatorname{div} \vec{F}}$$





$m = \frac{0-1}{2-0} = -\frac{1}{2}$

$y = -\frac{1}{2}x + 1$



Now, $x + 2y + z = 2$

$z = 2 - x - 2y$

$$\int_0^2 \int_0^{-\frac{1}{2}x+1} \int_0^{2-x-2y} 6xy\, dz\, dy\, dx = \int_0^2 \int_0^{-\frac{1}{2}x+1} \Big[6xyz\Big]_0^{2-x-2y} dy\, dx$$

$6xy(2-x-2y) = 12xy - 6x^2y - 12xy^2 = 12xy - 6x^2y - 12xy^2$

$$= \int_0^2 \int_0^{-\frac{1}{2}x+1} (12xy - 6x^2y - 12xy^2)\, dy\, dx = \int_0^2 \Big[6xy^2 - 3x^2y^2 - 4xy^3\Big]_0^{-\frac{1}{2}x+1} dx$$

$6x(-\frac{1}{2}x+1)^2 - 3x^2(-\frac{1}{2}x+1)^2 - 4x(-\frac{1}{2}x+1)^3$

$(-\frac{1}{2}x+1)^2 = \frac{1}{4}x^2 - x + 1$ TIMES $6x$, then Times $-3x^2$

$(-\frac{1}{2}x+1)(-\frac{1}{2}x+1)^2 = (-\frac{1}{2}x+1)(\frac{1}{4}x^2 - x + 1) = -\frac{1}{8}x^3 + \frac{1}{2}x^2 - \frac{1}{2}x = y^3$
$+\frac{1}{4}x^2 - x + 1$
$-\frac{1}{8}x^3 + \frac{3}{4}x^2 - \frac{3}{2}x + 1$

$6xy^2 = 6x(\frac{1}{4}x^2 - x + 1)$
$= \frac{3}{2}x^3 - 6x^2 + 6x$

$-3x^2y^2 = -3x^2(\frac{1}{4}x^2 - x + 1) = -\frac{3}{4}x^4 + 3x^3 - 3x^2$

$-4xy^3 = -4x\left[-\frac{1}{8}x^3 + \frac{1}{2}x^2 - \frac{1}{2}x\right] = \frac{1}{2}x^3 - 2x^2 + 2x$

$$= \int_0^2 \left[\frac{3}{2}x^3 - 6x^2 + 6x - \frac{3}{4}x^4 + 3x^3 - 3x^2 + \frac{1}{2}x^3 - 2x^2 + 2x\right] dx$$

$$\int_0^2 \left[-\frac{3}{4}x^4 + 5x^3 - 11x^2 + 8x\right] dx = \left[-\frac{3}{20}x^5 + \frac{5x^4}{4} - \frac{11}{3}x^3 + 4x^2\right]_0^2$$

$= \frac{-3}{20} \cdot 32 + \frac{5}{4} \cdot 16 - \frac{11}{3} \cdot 8 + 4 \cdot 4$

$= -\frac{24}{5} + 20 - \frac{88}{3} + 16$

$= \frac{(-24)(3) + (36)(15) - (88)(5)}{15}$

360
180
540

124
3
72

$= \frac{-72 + 540 - 440}{15} = \frac{540 - 512}{15} = \boxed{\frac{28}{15}}$

Assuming no

$$\int_0^2 \int_0^{-\frac{1}{2}\cdot x + 1} \int_0^{2 - x - 2\cdot y} 6 \cdot x \cdot y\, dz\, dy\, dx = \frac{2}{5}$$