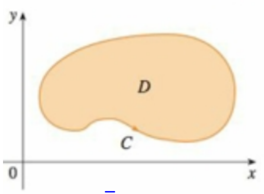
Test 5 & FINAL
Comprehensive Test 5 } Take-Home.
Due 12/11, Drop off → my mailbox. or scan to PDF.

Stokes adds little to the techniques & ideas of Green's, which is a special case of Stokes.

Stokes' Theorem.

We generalize Green's Theorem.

Recall from Section 16.4:



GREEN'S THEOREM Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$\vec F \cdot dr = \vec r'(t)\,dt$ $\qquad \vec r(t) = \langle x(t), y(t)\rangle$

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$\vec F = \langle P, Q\rangle$ $\quad dy\,dx$

$= \langle P(x(t),y(t)),\ Q(x(t),y(t))\rangle \cdot \langle x'(t), y'(t)\rangle\,dt$

I keep trying to express this as a "curl" sort of thing, bringing you guys back to the vector notation and the standard cross product we see, over and over, in the integrand.

$(P x'(t) + Q y'(t))\,dt$

$P = P(x,y) = P(x(t), y(t))$ $\qquad = P x'(t)\,dt + Q y'(t)\,dt$

$Q = Q(x,y) = Q(x(t), y(t))$ $\qquad = P\,dx + Q\,dy$

$\vec F = \langle P, Q, 0\rangle$

$\vec r = \vec r(t) = \langle x(t), y(t), 0\rangle \Rightarrow d\vec r = \langle x'(t), y'(t), 0\rangle\,dt$

$\mathrm{curl}(\vec F):\ \langle 0, 0, Q_x - P_y\rangle$

$$\begin{array}{l} \langle \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\rangle, \frac{d}{dx}, \frac{d}{dy} \\ \times \langle P, Q, 0\rangle, P, Q \\ \hline \langle 0, 0, Q_x - P_y\rangle \end{array}$$

$$\iint_{\mathcal{D}} \vec F \cdot d\vec S = \iint_{\mathcal{D}} \vec F \cdot \vec n\, dS \qquad \vec n = \frac{\vec r_u \times \vec r_v}{\|\vec r_u \times \vec r_v\|}$$

$$\int_C \vec F \cdot d\vec r = \iint_{\mathcal{D}} \mathrm{curl}\,\vec F \cdot d\vec S$$

$dS = \|\vec r_u \times \vec r_v\|$

$\iint \vec F \cdot (\vec r_u \times \vec r_v)$

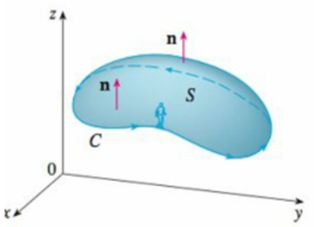


FIGURE 1

STOKES' THEOREM Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \mathrm{curl}\,\mathbf{F} \cdot d\mathbf{S}$$

$\mathrm{Curl}\,\vec F =$

It's hard to keep my mouth shut (so I don't) about the integral on the left, because I KNOW we wanted to state GREEN'S THEOREM in this language. Now this curl stuff requires a 3-D vector field, **F**. But we can make Green's work in 3-D just by adding a trivial 0 in the 3rd component of **F**.

Stokes' Theorem INCLUDES Green's Theorem as a special case! So it's this huge sledgehammer that covers everything.

$\vec T = \frac{\vec r'}{\|\vec r'\|}$ $\qquad ds = \|\vec r'\|\,dt$

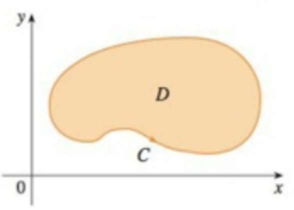
Since I don't know how to say this any better:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\,ds \quad \text{and} \quad \iint_S \mathrm{curl}\,\mathbf{F} \cdot d\mathbf{S} = \iint_S \mathrm{curl}\,\mathbf{F} \cdot \mathbf{n}\,dS$$

$\frac{\vec r_u \times \vec r_v}{\|\vec r_u \times \vec r_v\|}$ $\qquad \|\vec r_u \times \vec r_v\|\,du\,dv$

Stokes' Theorem says that the line integral around the boundary curve of S of the tangential component of $\mathbf{F}$ is equal to the surface integral of the normal component of the curl of $\mathbf{F}$.

$$\iint_S \mathrm{curl}\,\vec F \cdot (\vec r_u \times \vec r_v)\,du\,dv$$



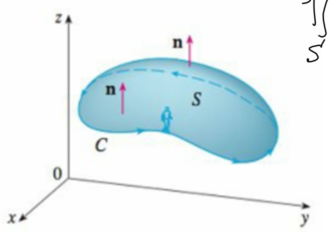


FIGURE 1

$$\int_C f(t)\,ds = \int_C f(t)\,\|\bar{r}'(t)\|\,dt$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$\int_C \bar{F}\cdot d\bar{r}$ = Line integral of tangential component of $\bar{F}$ along C.

STOKES : $\int_C \bar{F}\cdot d\bar{r} = \iint_D \text{curl}\,\bar{F}\,d\bar{S}$

C contains a domain D

$C = \partial D$ = Boundary of D.

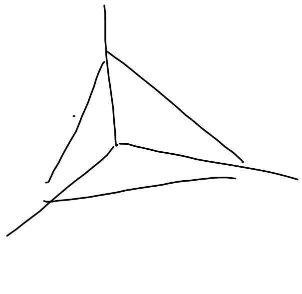
What's this $d\bar{S}$ as opposed to dS ?

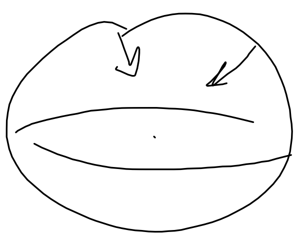
$$\downarrow$$

$$\bar{n}\,dS$$

$$\frac{\bar{r}_u\times\bar{r}_v}{\|\bar{r}_u\times\bar{r}_v\|}\,dS$$

$$\bar{r}_u\times\bar{r}_v\,du\,dv$$





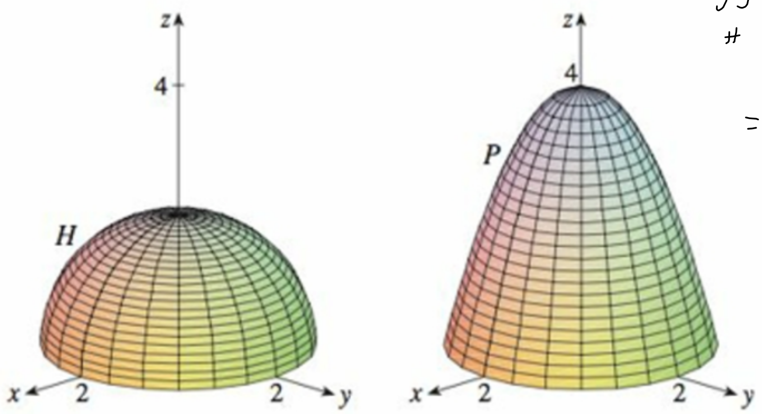
1. A hemisphere H and a portion P of a paraboloid are shown. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives. Explain why

$$\iint_H \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_P \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

$$\iint_H \text{curl}\bar{F} \cdot d\bar{S} = \int_C \bar{F} \cdot d\bar{r}$$

$$= \iint_P \text{curl}\bar{F} \cdot d\bar{S}, \text{ by}$$

Stokes' Thm.



2–6 Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

2. $\mathbf{F}(x, y, z) = 2y \cos z\, \mathbf{i} + e^x \sin z\, \mathbf{j} + xe^y\, \mathbf{k}$,
S is the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$, oriented upward

This exercise was worked in class. For the video, check out the ZOOM recording for May 1st in the Course Shell.

Spring '19!

$$\int_C \bar{F} \cdot d\bar{r} = \int_C \bar{F} \cdot \bar{T}\, ds = \int_C \bar{F} \cdot \bar{r}'\, dt$$

$$\bar{F} = \langle 2y\cos z, e^x \sin z, xe^y \rangle = \langle 2 \cdot 3\sin t \cos 0, e^x \sin 0, 3\cos t\, e^{3\sin t} \rangle$$

$$= \langle 6\sin t, 0, 3\cos t\, e^{3\sin t} \rangle$$

$$\bar{r}(t) = \langle 3\cos t, 3\sin t, 0 \rangle \qquad 0 \leq t \leq 2\pi$$

$$\bar{r}'(t) = \langle -3\sin t, 3\cos t, 0 \rangle$$

$$\bar{F} \cdot d\bar{r} = \bar{F}(\bar{r}(t)) \cdot \bar{r}'(t)\, dt$$

$$= \langle 6\sin t, 0, 3\cos t\, e^{3\sin t} \rangle \cdot \langle -3\sin t, 3\cos t, 0 \rangle\, dt$$

$$= \left(-18\sin^2 t + 0 + 0\right) dt = -18\left(\frac{1 - \cos(2t)}{2}\right) dt$$

$$= 9(\cos(2t) - 1)\, dt = (9\cos(t) - 9)\, dt$$

and $$\int_0^{2\pi} (9\cos t - 9)\, dt = \Big[9\sin t - 9t\Big]_0^{2\pi} = -18\pi$$

2–6 Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

3. $\mathbf{F}(x, y, z) = x^2z^2\,\mathbf{i} + y^2z^2\,\mathbf{j} + xyz\,\mathbf{k}$,
S is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$, oriented upward

$$\iint_S \operatorname{curl}\bar{F} \cdot \bar{n}\, dS = \cancel{\|\bar{r}_u \times \bar{r}_v\|}\, du\, dv$$

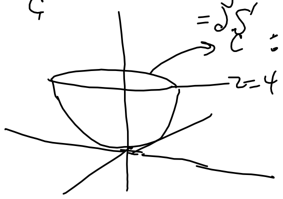
$$\iint_S \operatorname{curl}\bar{F} \cdot \frac{\bar{r}_u \times \bar{r}_v}{\|\bar{r}_u \times \bar{r}_v\|}$$

$$= \iint_S \operatorname{curl}(F(\bar{r}(u,v))) \cdot \bar{r}_u \times \bar{r}_v \, du\, dv$$

But we don't need all that, thx to Stokes!

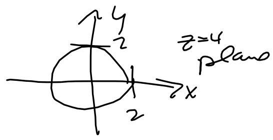
He says:

$$\iint_S \operatorname{curl}\bar{F} \cdot d\bar{S} = \int_C \bar{F} \cdot d\bar{r} = \int \bar{F}'(\bar{r}(t))\, \bar{r}'(t)\, dt$$



$= \partial S$: C: $\bar{r}(x,y) = \langle x, y, 4 \rangle$

$x^2+y^2=4$, so $x = 2\cos\theta$, $y = 2\sin\theta$, $z=4$



$0 \le \theta \le 2\pi$

$$\bar{r}(\theta) = \langle 2\cos\theta, 2\sin\theta, 4 \rangle$$

$$\boxed{\bar{r}'(\theta) = \langle -2\sin\theta, 2\cos\theta, 0 \rangle}$$

$$\bar{F}(\bar{r}(\theta)) = \langle (2^2\sin^2\theta)(4)^2, 2^2\cos^2\theta)(4)^2, (2\cos\theta)(2\sin\theta)(4) \rangle$$

$$\boxed{\bar{F}(\bar{r}(\theta)) = \langle 4^3\sin^2\theta, 4^3\cos^2\theta, 4^2\sin\theta\cos\theta \rangle}$$

$$\int_0^{2\pi} \left((4^3\sin^2\theta)(-2\sin\theta) + (4^3\cos^2\theta)(2\cos\theta) + 0\right) d\theta$$

$$128\int_0^{2\pi} \left(-\sin^3\theta + \cos^3\theta\right) d\theta =$$

Scratch $\sin^3\theta = \sin\theta(1-\cos^2\theta) = \sin\theta + \cos^2\theta(-\sin\theta)$

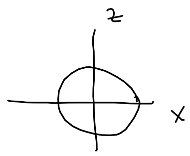
$\cos^3\theta = \cos\theta(1-\sin^2\theta) = \cos\theta - \sin^2\theta\cos\theta$

$$= 128\int_0^{2\pi} \left(-\sin\theta - \cos^2\theta(-\sin\theta) + \cos\theta - \sin^2\theta\cos\theta\right) d\theta$$

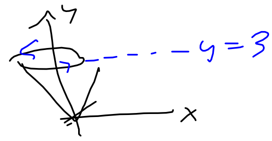
$$= 128\left[\cos\theta - \frac{\cos^3\theta}{3} + \sin\theta - \frac{\sin^3\theta}{3}\right]_0^{2\pi}$$

$$= 128\left[1 - \tfrac{1}{3} - \left(1 - \tfrac{1}{3}\right) + (0-0) - (0-0)\right] = 0$$

4. $\mathbf{F}(x, y, z) = x^2y^3z\,\mathbf{i} + \sin(xyz)\,\mathbf{j} + xyz\,\mathbf{k}$,
S is the part of the cone $y^2 = x^2 + z^2$ that lies between the planes $y = 0$ and $y = 3$, oriented in the direction of the positive y-axis



$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} \qquad C = \partial S$$



$$y = 3 \Rightarrow y^2 = 3^2 = x^2 + z^2 = 9$$

$$\vec{r}(\theta) = \langle 3\cos\theta, 3, 3\sin\theta \rangle$$
$$\vec{r}\,'(\theta) = \langle -3\sin\theta, 0, 3\cos\theta \rangle$$

$$\vec{F}(\vec{r}(\theta)) = \langle (3^2\cos^2\theta)(3^3)(3\sin\theta),\ \sin(3^3\cos\theta\sin\theta),\ 3^3\cos\theta\sin\theta \rangle$$

$$\int_{0 \to 2\pi}^{2\pi \to 0!} \left[(3^6\cos^2\theta\sin\theta)(-3\sin\theta) + 0 + (3^3\sin\theta\cos\theta)(3\cos\theta)\right] d\theta$$

CLOCKWISE!

$$= \int_0^{2\pi} \left[-3^7\cos^2\theta\sin^2\theta + 3^4\cos^2\theta\sin\theta\right] d\theta$$

Scratch: $\frac{1}{4}(1+\cos 2\theta)(1-\cos 2\theta)$

$$= \tfrac{1}{4}\left[1 - \cos^2 2\theta\right] = \tfrac{1}{4} - \tfrac{1}{4}\left[\tfrac{1}{2}(1 + \cos 4\theta)\right]$$
$$= \tfrac{1}{4} - \tfrac{1}{8} - \tfrac{1}{8}\cos 4\theta$$

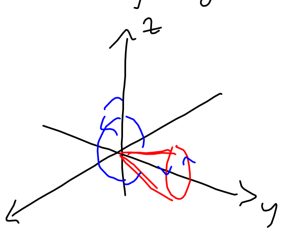
$$= -3^7 \cdot \frac{1}{8}\left[1 - \cos 4\theta\right]$$

$$= \frac{-3^7}{8}\int_0^{2\pi}(1 - \cos 4\theta)\,d\theta + 3^4\left[\frac{\cos^3\theta}{3}\right]_0^{2\pi}$$

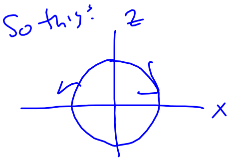
$$= -\frac{3^7}{8}\left[\theta\right]_0^{2\pi} - \frac{3^7}{8}\cdot\frac{1}{4}\int_0^{2\pi}\cos(4\theta)\cdot 4\,d\theta + 0$$

$$= -\frac{3^7}{8}\cdot 2\pi - \left[\frac{3^7}{32}\sin(4\theta)\right]_0^{2\pi} = -\frac{3^7}{4}\pi = \frac{-2187}{4}\pi$$

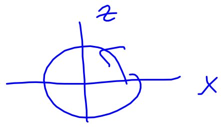
This is the wrong sign. Look at orientation



So this:



Not THIS:



Clockwise w/
$x = 3\cos\theta,\ z = 3\sin\theta$

So $\int_{2\pi}^{0}$ OR

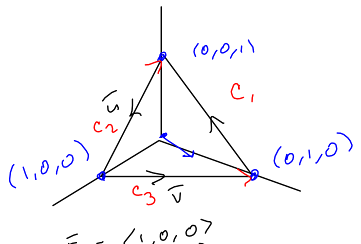
$\vec{r} = \langle 3\sin\theta, 0, 3\cos\theta \rangle$ would also do it for us.

§ 16.8

7–10 Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. In each case C is oriented counterclockwise as viewed from above.

Done in class. See Lecture Recording.

7. $\mathbf{F}(x, y, z) = (x + y^2)\,\mathbf{i} + (y + z^2)\,\mathbf{j} + (z + x^2)\,\mathbf{k}$,
C is the triangle with vertices (1, 0, 0), (0, 1, 0), and (0, 0, 1)



$\bar{r}_0 = \langle 1, 0, 0 \rangle$

$\int_C \bar{F} \cdot d\bar{r}$ Need to build 3 lines & eval 3 line integrals, w/o Stokes.

Eq'n of plane:

$\bar{u} = \langle -1, 0, 1 \rangle$, $-1, 0$

$\times$ $\bar{v} = \langle -1, 1, 0 \rangle$, $-1, 1$

$\langle -1, -1, -1 \rangle = \bar{n}$

$\bar{n} \cdot \langle x-1, y, z \rangle = 0$

$-1(x-1) - 1(y) - 1z = 0$

$-x + 1 - y - z = 0$

$z = 1 - x - y$ or $x + y + z = 1$

characterizes the surface

Now, $\iint_S \bar{F} \cdot d\bar{S} =$

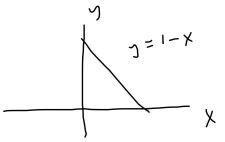
$\bar{r}$: $x = x,\ y = y,\ z = 1 - x - y$

$\bar{r}_x = \langle 1, 0, -1 \rangle$, $1, 0$

$\times$ $\bar{r}_y = \langle 0, 1, -1 \rangle$, $0, 1$

$\langle 1, 1, 1 \rangle = \bar{r}_x \times \bar{r}_y$

$\iint_{S'} \bar{F} \cdot \bar{n}\, dA =$



$$\int_0^1 \int_0^{1-x} (\text{curl } \bar{F}) \cdot \bar{r}_x \times \bar{r}_y \, dy\, dx$$

scratch: $\text{curl } \bar{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$

$\nabla \times \bar{F}$

$\times$ $\langle x + y^2, y + z^2, z + x^2 \rangle, x + y^2, y + z^2$

$\langle 2z, 2x, 2y \rangle = \text{curl } \bar{F}$

$$= \int_0^1 \int_0^{1-x} \langle 2z, 2x, 2y \rangle \cdot \langle 1, 1, 1 \rangle \, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \langle 2(1-x-y), 2x, 2y \rangle \cdot \langle 1, 1, 1 \rangle \, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} (2 - 2x - 2y + 2x + 2y)\, dy\, dx = \int_0^1\int_0^{1-x} 2\, dy\, dx = \int_0^1 \left[2y\right]_0^{1-x} dx$$

$$= 2 \int_0^1 (1-x)\, dx = 2\left[x - \tfrac{1}{2}x^2\right]_0^1 = 2\left[1 - \tfrac{1}{2}\right] = 1 \ !$$

= **STOKES' THEOREM** Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

$\bar{r}_x \times \bar{r}_y \, dy\, dx$

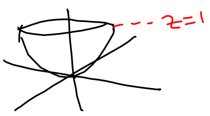
$$d\bar{S} = \bar{n}\, dS' = \frac{\bar{r}_x \times \bar{r}_y}{\|\bar{r}_x \times \bar{r}_y\|} \|\bar{r}_x \times \bar{r}_y\|\, dy\, dx$$

13–15 Verify that Stokes' Theorem is true for the given vector field **F** and surface S.

① $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ = ② $\int_C \vec{F}\cdot d\vec{r}$

13. $\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$,
S is the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$, oriented upward

①



$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2\rangle$

$\vec{F}(\vec{r},\theta) = \langle r^2\sin^2\theta, r\cos\theta, r^4\rangle$ is premature

$\nabla \times \vec{F}$

Need curl $\vec{F}$, $\vec{r}_r \times \vec{r}_\theta$

$$\begin{array}{c} \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \;\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \\ \times\; \langle y^2, x, z^2\rangle\; y^2, x \\ \hline \langle 0, 0, 1-1\rangle = \vec{0} \end{array}$$

$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_S \vec{0}\cdot d\vec{S} = 0$

$\langle y^2, x, z^2\rangle$

② $\int_C \vec{F}\cdot d\vec{r}$

$\vec{r}(\theta) = \langle \cos\theta, \sin\theta, 1\rangle$

$\vec{r}'(\theta) = \langle -\sin\theta, \cos\theta, 0\rangle$

$\vec{F}(\vec{r}(\theta)) = \langle \sin^2\theta, \cos\theta, 1\rangle$

$$\int_0^{2\pi} \left(-\sin^2\theta\cos\theta + \sin\theta\cos\theta\right) d\theta$$

$$= \left[-\frac{\sin^3\theta}{3} + \frac{\sin^2\theta}{2}\right]_0^{2\pi} = 0$$

15. $\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$,
S is the hemisphere $x^2 + y^2 + z^2 = 1$, $y \geq 0$, oriented in the direction of the positive y-axis

$y = \pm\sqrt{1-(x^2+z^2)} = \pm\sqrt{1-r^2}$ take $+$

$\iint_S \text{curl}\,\bar{F}\cdot d\bar{S} = \int_{C'} \bar{F}\cdot d\bar{r}$ ① ②

$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$

$\times \langle y, z, x \rangle\ y, z$

$\langle 0-1, -1, -1 \rangle = \boxed{\langle -1, -1, -1 \rangle = \text{curl}\,\bar{F}}$

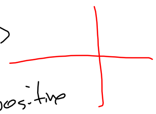
Due to orientation, the thing is going clockwise in the xz-plane, so 2π to 0 $\int_{2\pi}^{0}$

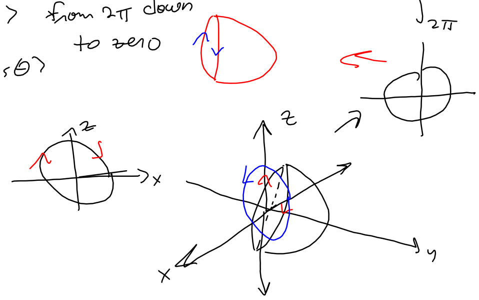
$\bar{r}(r,\theta) = \langle r\cos\theta, \sqrt{1-r^2}, r\sin\theta \rangle$ from 2π down to zero

$\times$ $\bar{r}_r = \langle \cos\theta, \frac{1}{2}(1-r^2)^{-\frac{1}{2}}(-2r), \cos\theta \rangle$

$= \left\langle \cos\theta, \frac{-r}{\sqrt{1-r^2}}, \sin\theta \right\rangle$

Better might be
$\left\langle r\sin\theta, \frac{-r}{\sqrt{1-r^2}}, r\cos\theta \right\rangle$
$0 \leq \theta \leq 2\pi$,
measured from positive z-axis





$\bar{r}_r = \left\langle \cos\theta, \frac{-r}{\sqrt{1-r^2}}, \sin\theta \right\rangle, \cos\theta, \frac{-r}{\sqrt{1-r^2}}$

$\times\ \bar{r}_\theta = \langle -r\sin\theta, 0, r\cos\theta \rangle, -r\sin\theta, 0$

$\left\langle \frac{-r^2\cos\theta}{\sqrt{1-r^2}}, -r\sin^2\theta - r\cos^2\theta, \frac{-r^2\sin\theta}{\sqrt{1-r^2}} \right\rangle$

$= \left\langle \frac{-r^2\cos\theta}{\sqrt{1-r^2}}, -r, \frac{-r^2\sin\theta}{\sqrt{1-r^2}} \right\rangle$

$\therefore \int_0^{2\pi}\int_0^1 \text{curl}\,\bar{F}\cdot \bar{r}_r \times \bar{r}_\theta\, dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 \langle -1,-1,-1 \rangle \cdot \left\langle \frac{-r^2\cos\theta}{\sqrt{1-r^2}}, -r, \frac{-r^2\sin\theta}{\sqrt{1-r^2}} \right\rangle$

$= \int_0^{2\pi}\int_0^1 \left(\frac{r^2\cos\theta}{\sqrt{1-r^2}} + r + \frac{r^2\sin\theta}{\sqrt{1-r^2}} \right) dr\,d\theta$

$= \int_0^{2\pi} (\cos\theta + \sin\theta)\,d\theta \int_0^1 \frac{r^2}{\sqrt{1-r^2}}\,dr + \int_0^{2\pi}\int_0^1 r\,dr\,d\theta$

$= \left[\sin\theta - \cos\theta\right]_0^{2\pi} (\qquad) + \int_0^{2\pi} \left[\frac{1}{2}r^2\right]_0^1 d\theta$

$= 0\ (\qquad) + \int_0^{2\pi} \frac{1}{2}\,d\theta$

$= 0 + \frac{1}{2}[2\pi] = \pi$

② $\int_{C'} \bar{F}\cdot d\bar{r} =$

$\bar{r}(\theta) = \langle \cos\theta, 0, \sin\theta \rangle$

$\bar{F} = \langle y, z, x \rangle = \langle 0, \sin\theta, \cos\theta \rangle$

$\bar{r}'(\theta) = \langle -\sin\theta, 0, \cos\theta \rangle$

$\bar{F}\cdot\bar{r}'(\theta) = 0 + 0 + \cos^2\theta$

$= \frac{1}{2} + \frac{1}{2}\cos(2\theta)$

~~If I'd done~~
~~$\bar{r}(\theta) = \langle \sin\theta, 0, \cos\theta \rangle$, $0 \leq \theta \leq 2\pi$~~
~~$\bar{r}'(\theta) = \langle \cos\theta, 0, -\sin\theta \rangle$~~
~~$\bar{F}(\bar{r}(\theta)) \cdot \bar{r}'(\theta)$~~
~~$= \langle 0, \cos\theta, \sin\theta \rangle \cdot \langle \cos\theta, 0, -\sin\theta \rangle$~~
~~$=$~~
~~$-\int_0^{2\pi}$~~

$\int_{2\pi}^{0} \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right) d\theta$

$= \left[\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right]_{2\pi}^{0}$

$= -\frac{1}{2}\cdot 2\pi = -\pi$

so I messed-up the orientation somehow.

16. Let C be a simple closed smooth curve that lies in the plane $x + y + z = 1$. Show that the line integral

$$\int_C z\,dx - 2x\,dy + 3y\,dz = \int_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\bar{F}\cdot d\bar{S}$$

$\langle z, -2x, 3y\rangle \cdot \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\rangle\, dt$

depends only on the area of the region enclosed by C and not on the shape of C or its location in the plane.

$\bar{F} = \langle z, -2x, 3y\rangle$

$\bar{r} = \langle x, y, 1-x-y\rangle$

$\bar{r}_x = \langle 1, 0, -1\rangle, 1, 0$

$\times\quad \bar{r}_y = \langle 0, 1, -1\rangle, 0, 1$

$\langle 1, 1, 1\rangle = \bar{r}_x \times \bar{r}_y$

Scratch: $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$

$\times \langle z, -2x, 3y\rangle, z, -2x$

$\langle 3, 1, -2\rangle = \text{curl}\bar{F}$

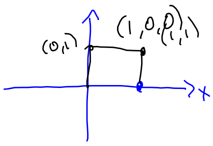
$$\iint_S \text{curl}\bar{F}\cdot d\bar{S} = \int_{x=a}^{x=b}\int_{y=c}^{y=d} 2\,dy\,dx = 2 \text{ Area of } S$$

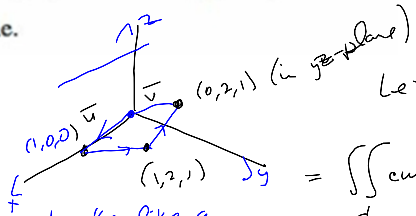
17. A particle moves along line segments from the origin to the points (1, 0, 0), (1, 2, 1), (0, 2, 1), and back to the origin under the influence of the force field

$$\mathbf{F}(x, y, z) = z^2\,\mathbf{i} + 2xy\,\mathbf{j} + 4y^2\,\mathbf{k}$$

Find the work done.

If I can avoid evaluating 4 line integrals, that'd be good!





Looks like a parallelogram.

Let's do Stokes to $\int_C \vec{F}\cdot d\vec{r}$

$= \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$

$\vec{r}$:

$\vec{u} = \langle 1,0,0\rangle, 1, 0$

$\times\ \vec{v} = \langle 0,2,1\rangle, 0, 2$

$\langle 0,-1,2\rangle = \vec{n}$ not unit vector

pick $(1,0,0)$ & (x,y,z) in plane.

Then $\vec{n}\cdot\langle x-1, y, z\rangle = 0$

$\langle 0,-1,2\rangle\cdot\langle x-1,y,z\rangle = 0$

$-y+2z = 0$

$y = 2z$, so

$\vec{r} = \langle x, 2z, z\rangle$

Parametric surface way

$\vec{r}(s,t) = \langle 1,0,0\rangle + s\langle 1,0,0\rangle + t\langle 0,2,1\rangle$

$= \langle 1+s, 2t, t\rangle$

$x = 1+s$
$s = x-1$
$t = z$

$z=0 \Rightarrow t=-1$
$z=1 \Rightarrow t=0$

$z=0$ to $z=1$

$\vec{F} = \langle z^2, 2xy, 4y^2\rangle$

$\vec{F} = \langle z^2, 2x(2z), 4(2z)^2\rangle = \langle z^2, 4xz, 16z^2\rangle$

$\vec{r}_x = \langle 1,0,0\rangle, 1, 0$

$\times\ \vec{r}_z = \langle 0,2,1\rangle, 0, 2$

$\langle 0,-1,2\rangle = \vec{r}_x\times\vec{r}_z$

curl $\vec{F}$:

$\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$

$\times\ \langle z^2, 4xz, 16z^2\rangle, z^2, 4xz$

$\langle -4x, 2z, 4z\rangle = \operatorname{curl}\vec{F}$

$\operatorname{curl}\vec{F}\cdot\vec{r}_x\times\vec{r}_z = 0 - 2z + 8z = 6z$

$\iint_S \cdots = \int_0^1\int_0^1 6z\,dz\,dx = 6\int_0^1 dx \int_0^1 z\,dz$

$= 6\left[x\right]_0^1\left[\tfrac{1}{2}z^2\right]_0^1 = 6[1]\left[\tfrac{1}{2}\right]$

$\boxed{= 3}$

Using $\vec{r}(s,$

$\vec{r}(s,t) = \langle 1+s, 2t, t\rangle$

$\vec{r}_s = \langle 1,0,0\rangle, 1, 0$

$\times\ \vec{r}_t = \langle 0,2,1\rangle, 0, 2$

$\langle 0,-1,2\rangle = \vec{r}_s\times\vec{r}_t$

$\langle -4x, 2z, 4z\rangle = \operatorname{curl}\vec{F}$

$\Rightarrow \operatorname{curl}\vec{F}(s,t) = \langle -4-4s, 2t, 4t\rangle$

$\operatorname{curl}\vec{F}\cdot\vec{r}_s\times\vec{r}_t = 0 - 2t + 8t = 6t$

$\int\int(6t)\,ds\,dt$

$= \int_{-1}^0\int_0^1 6t\,ds\,dt$

$= \left[\int_{-1}^0 6t\right]\left[\int_1^0 ds\right]$

$= \left(\left[3t^2\right]_0^1\right)\left(\left[s\right]_{-1}^0\right)$

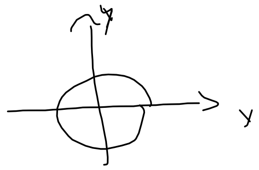
$= (3-0)(0-(-1)) = 3$

18. Evaluate $\qquad \bar{F} = \langle y+\sin x,\ z^2+\cos y,\ x^3 \rangle$

$$\int_C (y + \sin x)\,dx + (z^2 + \cos y)\,dy + x^3\,dz$$

where C is the curve $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle$, $0 \le t \le 2\pi$. [*Hint:* Observe that C lies on the surface $z = 2xy$.]

$z = 2xy$



$\bar{r}(t) = \langle r\sin t,\ r\cos t,\ 2r^2\sin t\cos t \rangle$

$1 - \sin^2 t - \sin^2 t$
$= 1 - 2\sin^2 t$

$\bar{r}_r = \langle \sin t,\ \cos t,\ 2r\sin t\cos t \rangle$

$\bar{r}_t = \langle r\cos t,\ -r\sin t,\ 2r^2(\cos^2 t - \sin^2 t) \rangle$

$= \langle r\cos t,\ -r\sin t,\ 2r^2 - 4r^2\sin^2 t \rangle$

$\bar{r}_r = \langle \sin t,\ \cos t,\ 2r\sin t\cos t \rangle,\ \sin t,\ \cos t$

$\bar{r}_t = \langle r\cos t,\ -r\sin t,\ 2r^2(1 - 2\sin^2 t) \rangle,\ r\cos t,\ -r\sin t$

$\langle 2r^2\cos t(1 - 2\sin^2 t) + 2r^2\sin^2 t\cos t,\ 2r^2\cos^2 t\sin t,\ -r\sin^2 t - r\cos^2 t \rangle$

$= \langle 2r^2\cos t - 2r^2\sin^2 t,\ 2r^2\cos^2\sin t,\ -r \rangle = \bar{r}_r \times \bar{r}_t$

$\bar{F} = \langle y+\sin x,\ z^2+\cos y,\ x^3 \rangle$

$\text{curl } \bar{F} = \left\langle \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z} \right\rangle,\ \frac{\partial}{\partial x},\ \frac{\partial}{\partial y}$

$\times \quad y+\sin x,\ z^2+\cos y,\ x^3,\ y+\sin x,\ z^2+\cos y$

$\langle 0 - 2z,\ 3x^2,\ 1 \rangle = \text{curl } \bar{F}$

$\text{curl } \bar{F} = \boxed{\langle -2(2r^2\sin t\cos t)},\ 3r^2\sin^2 t,\ 1 \rangle$

$\boxed{\langle 2r^2\cos t - 2r^2\sin^2 t},\ 2r^2\cos^2\sin t,\ -r \rangle = \bar{r}_r \times \bar{r}_t$

$= \displaystyle\int_0^{2\pi}\int_0^1$

19. If S is a sphere and $\mathbf{F}$ satisfies the hypotheses of Stokes' Theorem, show that $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$. $\bar F$ is twice dif'bl

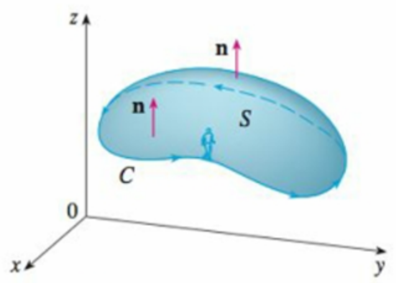
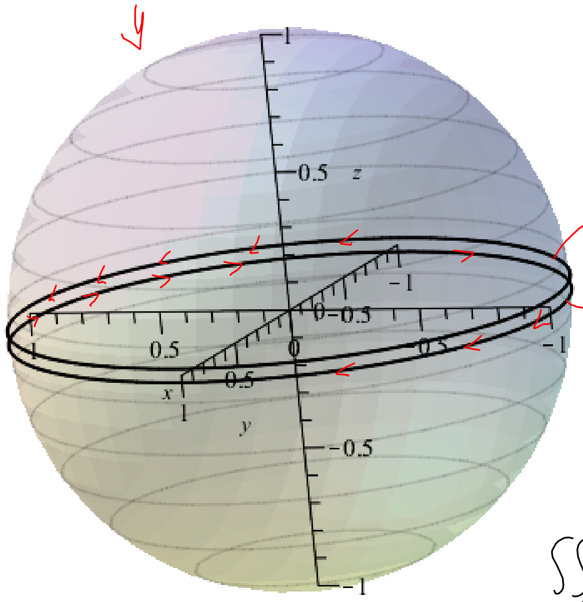


FIGURE 1

STOKES' THEOREM Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

$$\iint_{S'} = \iint_{S_1'} + \iint_{S_2'}$$



S' = sphere outward

S_1' = Top half upward

S_2' = Bottom half, downward

Looking down, we traverse C_1 counter clockwise

Traverse C_2, looking up make it counterclockwise.

$$\iint_{S'} \operatorname{curl} \bar F \cdot d\bar S = \sum_{k=1}^{2} \iint_{S_k'} \operatorname{curl} \bar F \cdot d\bar S_k$$

$$= \iint_{S_1'} \operatorname{curl} \bar F \cdot d\bar S_1 + \iint_{S_2'} \operatorname{curl} \bar F \cdot d\bar S_2 = \int_{C_1} \bar F \cdot d\bar r + \int_{C_2} \bar F \cdot d\bar r$$

$C_2 = -C_1 \Longrightarrow$ the previous is $\int_{C_1} \bar F \cdot d\bar r + \int_{-C_1} \bar F \cdot d\bar r$

$$= \int_{C_1} \bar F \cdot d\bar r - \int_{C_1} \bar F \cdot d\bar r = 0$$

20. Suppose S and C satisfy the hypotheses of Stokes' Theorem and f, g have continuous second-order partial derivatives. Use Exercises 24 and 26 in Section 16.5 to show the following.

(a) $\int_C (f\,\nabla g) \cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \cdot d\mathbf{S}$

(b) $\int_C (f\,\nabla f) \cdot d\mathbf{r} = 0$

(c) $\int_C (f\,\nabla g + g\nabla f) \cdot d\mathbf{r} = 0$

Not too interested in your doing this. It's all good, but we're on a time line, here.

We did put some time in on these @ the end of 16.5.

Last mod: 4-D! $\langle x, y, z, t\rangle$

$\bar{r}'(t)$

$\frac{d\bar{r}}{dt}$

$$\int_C f\,\nabla g \cdot d\bar{r} = \iint_S \text{curl}(f\,\nabla g)\cdot d\bar{S}$$

$f = (x, y, z)$, $g = (u, v, w)$

$f\,\nabla g = \langle f u_x, f v_y, f w_z\rangle$

$\text{curl}(f\,\nabla g) = \nabla \times f\,\nabla g$

$$\left\langle \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right\rangle \frac{d}{dx}, \frac{d}{dy}$$

$$= \times \langle f u_x, f v_y, f w_z\rangle\, f u_x, f v_y$$

$$\left\langle \frac{d}{dy}(f w_z) - \frac{d}{dz}(f v_y), \frac{d}{dz}(f u_x) - \frac{d}{dx}(f w_z), \frac{d}{dx}(f v_y) - \frac{d}{dy}(f u_x)\right\rangle$$

$$= f_y w_z + f w_{zy} - f_z v_y - f v_{yz}, f_z u_x + f u_{xz} - f_x w_z - f w_{zx}, f_x v_y + f v_{yx} - f_y u_x - f u_{xy} \rangle$$

§16.5

$$\int_C \bar{F}\cdot\bar{n}\,ds = \iint_D \operatorname{div}\bar{F}\,dA \rightarrow §16.9 \rightarrow \iint_D \bar{F}\cdot\bar{n}\,ds = \iiint_E \operatorname{div}\bar{F}\,dV$$

Divergence Theorem.

In 2-D: $\bar{n} = \bar{k}$

$\bar{r} = \langle P, Q \rangle$

outward normal: $\bar{n} = \left\langle \frac{y'(t)}{\|\bar{r}'(t)\|}, \frac{-x'(t)}{\|\bar{r}'(t)\|} \right\rangle$ is clearly $\perp$ to $\bar{T}$

$-\bar{n} = \bar{N}$ from §13.2 $\bar{N} = \frac{\bar{T}'}{\|\bar{T}'\|}$ is INWARD NORMAL.

$\boxed{\text{in 2-D}}$

$\bar{n} \perp \bar{T}$ $\quad \frac{1}{\|\bar{r}'(t)\|^2}\langle x', y'\rangle \cdot \langle y', -x'\rangle = 0$

$x'y' - x'y'$!

dS -vs- $d\bar{S}$

§16.6? §16.7?