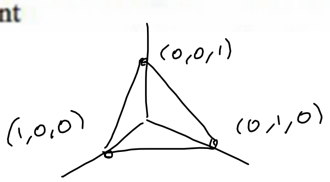
§16.7 Surface integrals.

5–18 Evaluate the surface integral.

7. $\iint_S yz\, dS$,
S is the part of the plane $x + y + z = 1$ that lies in the first octant



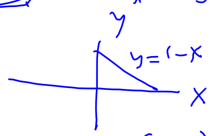
$$\bar{r} = \langle x, y, 1-x-y \rangle$$

$$\bar{r}_x = \langle 1, 0, -1 \rangle \quad 1, 0$$

$$\times \quad \bar{r}_y = \langle 0, 1, -1 \rangle \quad 0, 1$$

$$\langle 1, 1, 1 \rangle$$

$$\Rightarrow \quad \|\bar{r}_x \times \bar{r}_y\| = \sqrt{3}$$



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$$yz = y(1-x)$$

$$\iint_S yz\, dS \qquad \int_0^1 \int_0^{1-x} y(1-x-y)\sqrt{3}\, dy\, dx$$

$$\sqrt{3}\int_0^1 \int_0^{1-x} \left[y - xy - y^2 \right] dy\, dx = \sqrt{3} \int_0^1 \left[\tfrac{1}{2}y^2 - \tfrac{1}{2}y^2 x - \tfrac{1}{3}y^3 \right]_0^{1-x} dx$$

$$= \frac{\sqrt{3}}{2} \int_0^1 \left((1-x)^2 - x(1-x)^2 \right) dx - \frac{\sqrt{3}}{3} \int_0^1 (1-x)^3\, dx$$

$$= \frac{\sqrt{3}}{2} \int_0^1 \left(x^2 - 2x + 1 - x^3 + 2x^2 - x \right) dx + \frac{\sqrt{3}}{3} \left[\frac{(1-x)^4}{4} \right]_0^1$$

$$= \frac{\sqrt{3}}{2} \int_0^1 \left(-x^3 + 3x^2 - 3x + 1 \right) dx + \frac{\sqrt{3}}{3} \left[-\frac{1}{4} \right]$$

$$= \frac{\sqrt{3}}{2} \left[-\frac{x^4}{4} + x^3 - \frac{3}{2}x^2 + x \right]_0^1 - \frac{\sqrt{3}}{12}$$

$$= \frac{\sqrt{3}}{2} \left[-\frac{1}{4} + 1 - \frac{3}{2} + 1 \right] - \frac{\sqrt{3}}{12}$$

$$= \frac{\sqrt{3}}{2} \left[2 - \frac{7}{4} \right] - \frac{\sqrt{3}}{12} = -\frac{\sqrt{3}}{2} \left[\frac{8-7}{4} \right] - \frac{\sqrt{3}}{12}$$

$$= \frac{+\sqrt{3}}{8} - \frac{\sqrt{3}}{12} = \frac{+3\sqrt{3}}{24} - \frac{2\sqrt{3}}{24} = \boxed{\frac{\sqrt{3}}{24}}$$

9. $\iint_S yz\,dS$,
S is the surface with parametric equations $x = u^2$, $y = u\sin v$, $z = u\cos v$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \pi/2$

$\bar{r} = \langle u^2, u\sin v, u\cos v \rangle$ $\qquad$ $u =$ constant

$\langle 1, \sin v, \cos v \rangle$

$\bar{r}_u = \langle 2u, \sin v, \cos v \rangle$ $\quad 2u, \sin v$

$\times \quad \bar{r}_v = \langle 0, u\cos v, -u\sin v \rangle$ $\quad 0, u\cos v$

$\langle -u\sin^2 v - u\cos^2 v, 2u^2 \sin v, 2u^2\cos v \rangle$

$= \langle -u, 2u^2 \sin v, 2u^2 \cos v \rangle$

$\| \bar{r}_u \times \bar{r}_v \| = \sqrt{u^2 + 4u^4 \sin^2 v + 4u^4 \cos^2 v} = \sqrt{u^2 + 4u^4}$

$= u\sqrt{1+4u^2} \quad \cancel{= \frac{1}{4}\left(4u\sqrt{1+4u^2}\right)}$

$$\iint_S yz\,dS = \iint_S (u\sin v)(u\cos v)\left(u\sqrt{1+4u^2}\right) du\,dv$$

$$= \int_0^{\frac{\pi}{2}} \sin v \cos v\,dv \int_0^1 u^3 \sqrt{1+4u^2}\,du$$

$dv = 8u(1+4u^2)^{\frac{1}{2}}$ $\qquad$ $u = \frac{1}{8}u^2$

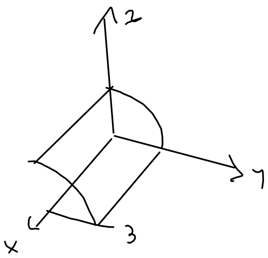
$v = \frac{2(1+4u^2)^{\frac{3}{2}}}{3}$ $\qquad$ $du = \frac{1}{4}u\,du$

$uv - \int v\,du = \frac{1}{8}u^2 \left(\frac{2(1+4u^2)^{\frac{3}{2}}}{3} \right)$

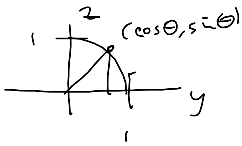
$- \frac{1}{4}\int \frac{2}{3}(1+4u^2)^{\frac{3}{2}} (u\,du)$

14. $\iint_S y^2 \, dS$,
S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane

17. $\iint_S (z + x^2 y)\, dS$,
S is the part of the cylinder $y^2 + z^2 = 1$ that lies between the planes $x = 0$ and $x = 3$ in the first octant



$x = x,\ y = \cos\theta,\ z = \sin\theta$



$0 \le \theta \le \frac{\pi}{2},\ 0 \le x \le 3$

$\bar{r} = \langle x, \cos\theta, \sin\theta \rangle$

$\bar{r}_x = \langle 1, 0, 0 \rangle,\ 1,\ 0$

$\times\ \bar{r}_\theta = \langle 0, -\sin\theta, \cos\theta \rangle,\ 0,\ -\sin\theta$

$\langle 0, -\cos\theta, -\sin\theta \rangle$

$\Rightarrow \|\bar{r}_x \times \bar{r}_\theta\| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

$\iint_S (z + x^2 y)\, dS$

$$\iint_S (z + x^2 y)\, dS = \int_0^3 \int_0^{\frac{\pi}{2}} (\sin\theta + x^2\cos\theta)\, d\theta\, dx$$

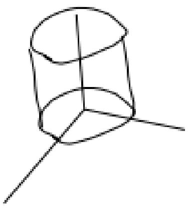
$$= \int_0^3 \left[-\cos\theta + x^2\sin\theta\right]_0^{\frac{\pi}{2}} dx$$

$$= \int_0^3 (1 + x^2)\, dx = \left[x + \frac{x^3}{3}\right]_0^3 = 3 + 9 = \boxed{12}$$

41. A fluid has density 870 kg/m^3 and flows with velocity $\mathbf{v} = z\,\mathbf{i} + y^2\,\mathbf{j} + x^2\,\mathbf{k}$, where x, y, and z are measured in meters and the components of $\mathbf{v}$ in meters per second. Find the rate of flow outward through the cylinder $x^2 + y^2 = 4$, $0 \leq z \leq 1$.

$$\int_S \bar{F}\cdot d\bar{S} = \int_S \bar{F}\cdot\bar{n}\,dS = \int_{S'} \bar{F}\cdot(\bar{r}_u \times \bar{r}_v)\,dA$$

$\bar{v} = \langle z, y^2, x^2\rangle$, $x, y, z \sim m$, $\bar{v} \sim m/s$, $S = \{(x,y,z) \mid x^2+y^2=4,\ 0 \le z \le 1\}$



S_1 = Bottom, S_2 = Top, S_3 = sides

Bottom: $\bar{n}_1 = \langle 0,0,-1\rangle$ (But let's work it out like a machine.

$0 \le r \le 2$, $0 \le \theta \le 2\pi$, $z = 0$!

$$\bar{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 0\rangle$$
$$\bar{r}_r = \langle \cos\theta, \sin\theta, 0\rangle \quad \cos\theta, \sin\theta$$
$$\times\ \bar{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle \quad -r\sin\theta, r\cos\theta$$
$$= \langle 0, 0, r\cos^2\theta + r\sin^2\theta\rangle$$
$$= \langle 0,0,r\rangle \text{ But oriented down.}$$

$\bar{v} = \langle z, y^2, x^2\rangle$

$= \langle 0, r^2\sin^2\theta, r^2\cos^2\theta\rangle$

$\bar{v}\cdot(\bar{r}_r \times \bar{r}_\theta) =$

$\langle 0, r^2\sin^2\theta, r^2\cos^2\theta\rangle \cdot \langle 0,0,r\rangle = -r^3\cos^2\theta$

S_1 So, $-\rho \int_0^{2\pi}\int_0^2 r^3\cos^2\theta\,dr\,d\theta = -\rho\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta \int_0^2 r^3\,dr$ etc.

S_2 : $\bar{r}_2(r,\theta) = \langle r\cos\theta, r\sin\theta, 1\rangle$

$$\bar{r}_r = \langle \cos\theta, \sin\theta, 0\rangle \quad \cos\theta, \sin\theta$$
$$\times\ \bar{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle \quad -r\sin\theta, r\cos\theta$$
$$\langle 0,0, r\cos^2\theta + r\sin^2\theta\rangle = \langle 0,0,r\rangle \text{ up}$$

$\bar{v} \cdot (\bar{r}_r \times \bar{r}_\theta) = \langle 0, r^2\sin^2\theta, r^2\cos^2\theta\rangle \cdot \langle 0,0,r\rangle$
$= r^3\cos^2\theta$

S_2: So $\displaystyle \int_0^{2\pi}\int_0^2 r^3\cos^2\theta \, dr\, d\theta$

$S_1 + S_2 = 0$! (opposite signs from opposite orientation

S_3 sides : $\bar{r}_3(\theta, z) = \langle 2\cos\theta, 2\sin\theta, z\rangle$

$\bar{r}_{3\theta} = \langle -2\sin\theta, 2\cos\theta, 0\rangle$ $-2\sin\theta, 2\cos\theta$

$\bar{r}_{3z} = \langle 0, 0, 1,$ $0, 0$

$\langle 2\cos\theta, 2\sin\theta, 0\rangle = \bar{r}_{3\theta} \times \bar{r}_{3z}$

$\bar{v}(\theta, z)$
$= \langle z, 4\sin^2\theta, 4\cos^2\theta\rangle$ $\bar{v} = \langle z, y^2, x^2\rangle$

$\bar{v} \cdot (\bar{r}_{3\theta} \times \bar{r}_{3z}) = \langle z, 4\sin^2\theta, 4\cos^2\theta\rangle \cdot \langle 2\cos\theta, 2\sin\theta, 0\rangle$

$= 2z\cos\theta + 8\sin^2\theta\cos\theta + 0$ $dz\,d\theta$

So, $\displaystyle \int_0^1\int_0^{2\pi} (2z\cos\theta + 8\sin^2\theta\cos\theta)\, d\theta\, dz = 0$

So Flux is zero!

$\displaystyle = \int_0^1 \left[2z\sin\theta + 8\frac{\sin^3\theta}{3}\right]_0^{2\pi} dz = \int_0^1 0\, dz = 0$