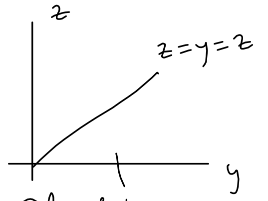
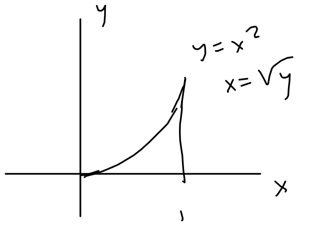
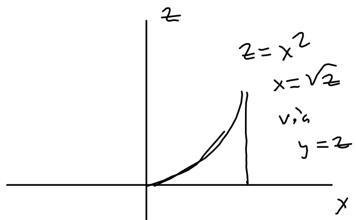
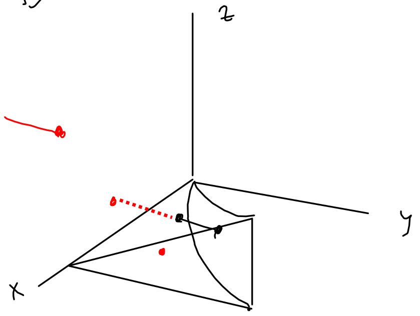
1. (10 pts) Write 5 other iterated integrals that are equivalent to the iterated integral $\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)\,dz\,dy\,dx$



Plane!







Outside-in to build the iterates

$$\int_0^1\int_{\sqrt{y}}^1\int_0^y f\,dz\,dx\,dy$$

$$\int_0^1\int_{\sqrt{z}}^1\int_z f\,dy\,dx\,dz$$

We extend what we did for parametric curves in the plane and in space to surfaces defined over plane regions. We had just 1 parameter, t, and now we have 2 parameters, u and v. Now, **r** is a surface.

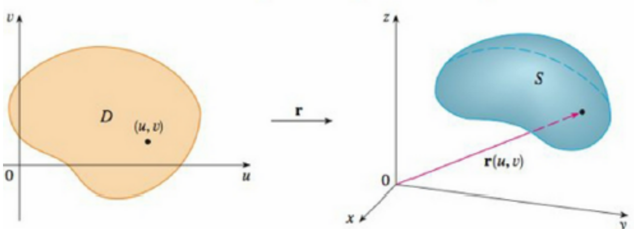
linear combo of $\vec{i}, \vec{j}, \vec{k}$

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} = \langle x(u,v), y(u,v), z(u,v) \rangle$$

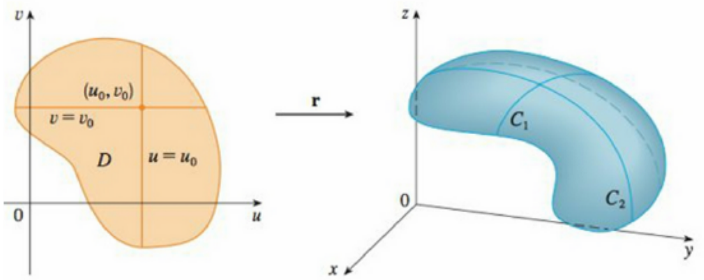
$$x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

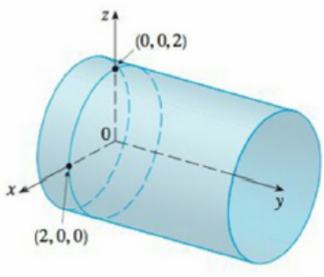
These are the parametric equations for the curve.

And if you remember what we did in the last chapter, what we do, here is virtually the same ideas that we used in the theory of change-of-variable in multiple integrals. You'll find yourself waiting to hear "Jacobian" but you never quite do...



Grid Lines: Hold one of u or v constant, and the result is a curve embedded in the surface on the right, such as C_1 or C_2. I don't think there's any way of telling which of the two curves corresponds to which of the grid lines in D, just from the general picture.





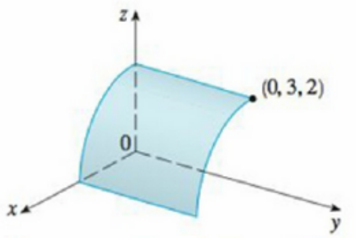
EXAMPLE 1 Identify and sketch the surface with vector equation

$\langle 2\cos u, v, 2\sin u\rangle = \quad \mathbf{r}(u, v) = 2\cos u\,\mathbf{i} + v\,\mathbf{j} + 2\sin u\,\mathbf{k}$

$$x = 2\cos u \qquad y = v \qquad z = 2\sin u$$

So for any point (x, y, z) on the surface, we have

$$x^2 + z^2 = 4\cos^2 u + 4\sin^2 u = 4$$



$0 \leqslant u \leqslant \pi/2 \qquad 0 \leqslant v \leqslant 3$

Restricted cylinder

EXAMPLE 3 Find a vector function that represents the plane that passes through the point P_0 with position vector $\mathbf{r}_0$ and that contains two nonparallel vectors $\mathbf{a}$ and $\mathbf{b}$.

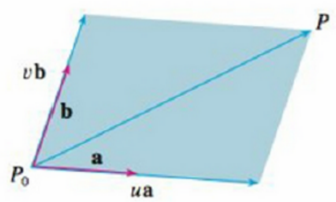
$$\mathbf{r}(u, v) = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$$

$P_0(x_0, y_0, z_0) \longleftrightarrow \vec{r}_0 = \langle x_0, y_0, z_0 \rangle$

You can navigate to any point in the plane by starting at $\mathbf{r}_0$ and moving some distance in the direction of $\mathbf{a}$ plus some distance in the direction of $\mathbf{b}$.

$\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3$

$\vec{r}(u,v) = \langle x_0, y_0, z_0 \rangle + u \langle a_1, a_2, a_3 \rangle + v \langle b_1, b_2, b_3 \rangle$



$x = x_0 + ua_1 + vb_1$

$y = y_0 + ua_2 + vb_2$

$z = z_0 + ua_3 + vb_3$

EXAMPLE 4 Find a parametric representation of the sphere

$$x^2 + y^2 + z^2 = a^2$$

The sphere has a simple representation $\rho = a$ in spherical coordinates

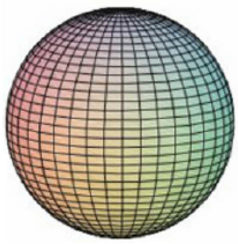
$$x = a \sin\phi \cos\theta \qquad y = a \sin\phi \sin\theta \qquad z = a \cos\phi$$

The corresponding vector equation is

$$\mathbf{r}(\phi, \theta) = a \sin\phi \cos\theta\, \mathbf{i} + a \sin\phi \sin\theta\, \mathbf{j} + a \cos\phi\, \mathbf{k}$$

We have $\boxed{0 \le \phi \le \pi \text{ and } 0 \le \theta \le 2\pi,}$ so the parameter domain is the rectangle $D = [0, \pi] \times [0, 2\pi]$. The grid curves with ϕ constant are the circles of constant latitude (including the equator). The grid curves with θ constant are the meridians (semicircles), which connect the north and south poles.

We already did this in Spherical Coordinates section AND we hit it, again, in the Change-Of-Variables section, where it was a handy example.



$\vec{r}(\phi, \theta) = \langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi \rangle$

$\phi = \frac{\pi}{2}$ = equator

$\phi = \frac{\pi}{6}$ = arctic circle

$\phi = \frac{5\pi}{6}$ = antarctic circle.

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\frac{x}{a}=\cos\theta$, $\frac{y}{b}=\sin\theta$

$d\bar{S}$

EXAMPLE 6 Find a vector function that represents the elliptic paraboloid $z = x^2 + 2y^2$.

SOLUTION If we regard x and y as parameters, then the parametric equations are simply

$$x = x \qquad y = y \qquad z = x^2 + 2y^2$$

and the vector equation is $\bar{r} = \langle x, y, x^2+2y^2 \rangle$

$$\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + (x^2 + 2y^2)\,\mathbf{k}$$

$\bar{r}_x = \langle 1, 0, 2x \rangle$

$\bar{r}_y = \langle 0, 1, 4y \rangle$

dS = increment of area on the surface $z = x^2+2y^2$ = dA in xy-plane. We want the corresponding area on the surface.

EXAMPLE 7 Find a parametric representation for the surface $z = 2\sqrt{x^2 + y^2}$, that is, the top half of the cone $z^2 = 4x^2 + 4y^2$.

SOLUTION 1 One possible representation is obtained by choosing x and y as parameters:

$$x = x \qquad y = y \qquad z = 2\sqrt{x^2 + y^2}$$

$\bar{r} = \bar{r}(x,y) = \langle x, y, 2\sqrt{x^2+y^2} \rangle$

$$\mathbf{r}(r, \theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + 2r\,\mathbf{k} = \langle r\cos\theta, r\sin\theta, 2r \rangle$$

where $r \geq 0$ and $0 \leq \theta \leq 2\pi$.

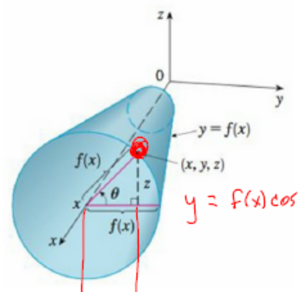
Not really a whole lot we haven't seen. We're just bringing it together and building some muscles for what's to come.

So, surface area $A(S')$ for $\boxed{E6}$ would be

$$\iint_R 1\, dS \qquad dS' = \|\bar{r}_x \times \bar{r}_y\|\,\underline{dy\,dx}$$

Increment area on of surface

Area in the parameter domain.



$y = f(x)\cos\theta$

y $f(x)$

SURFACES OF REVOLUTION

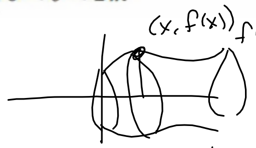
Surfaces of revolution can be represented parametrically and thus graphed using a computer. For instance, let's consider the surface S obtained by rotating the curve $y = f(x)$, $a \leqslant x \leqslant b$, about the x-axis, where $f(x) \geqslant 0$. Let θ be the angle of rotation as shown in Figure 9. If (x, y, z) is a point on S, then

$$x = x \qquad y = f(x)\cos\theta \qquad z = f(x)\sin\theta \tag{3}$$

Therefore we take x and θ as parameters and regard Equations 3 as parametric equations of S. The parameter domain is given by $a \leqslant x \leqslant b$, $0 \leqslant \theta \leqslant 2\pi$.

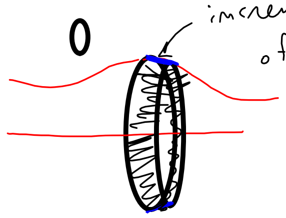
Rotation about x-axis

S



$f(x) = \text{radius}$

Circumference $= 2\pi r = 2\pi f(x)$
$= 2\pi y$



increment of arc length

To get surface area we multiply that circumference by ... increment of arc length

$\int 2\pi y \, ds$

$2\pi \int f(x) \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2} \, dx$

$= 2\pi \int f(x) \boxed{\underbrace{\sqrt{1 + f'(x)^2} \, dx}_{ds}}$

$\bar{r}(x, \theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta \rangle$

$\bar{r}_x = \langle 1, f'(x)\cos\theta, f'(x)\sin\theta \rangle$

$\bar{r}_\theta = \langle 0, -f(x)\sin\theta, f(x)\cos\theta \rangle$

Now we have an increment of area dS

$\bar{r}_x \times \bar{r}_\theta$

$\langle 1, f'\cos\theta, f'\sin\theta \rangle, 1, f'\cos\theta$

$\langle 0, -f\sin\theta, f\cos\theta \rangle, 0, -f\sin\theta$

$\langle ff'\cos^2\theta + ff'\sin^2\theta, -f\cos\theta, -f\sin\theta \rangle$

$= \langle ff', -f\cos\theta, -f\sin\theta \rangle$

$\| \vec{n} \| = \sqrt{(ff')^2 + f^2\cos^2 + f^2\sin^2\theta}$

$= \sqrt{(ff')^2 + f^2}$

$= f\sqrt{f'^2 + 1}$

$= f(x)\sqrt{f'(x)^2 + 1}$

$=$

in 1-D

$= \sqrt{1 + f'(x)^2} \, dx$

$\longleftrightarrow \|\bar{r}_x \times \bar{r}_y\| \, dy \, dx$ in 2-D.

So, $A(s) = \int 2\pi y \, ds$

$= \iint_R dS = \iint_R \|\bar{r}_u \times \bar{r}_v\| \, dA$

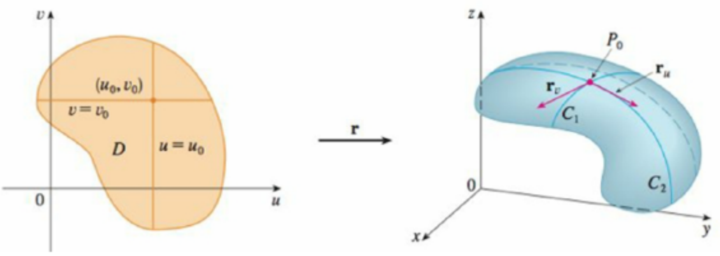
increment of area on surface

increment of area in the parameter domain.

Tangent Planes. Not really much new, here.

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\,\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\,\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\,\mathbf{k} \qquad \mathbf{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\,\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\,\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\,\mathbf{k}$$



Throwing this in, but it should be natural to you, by now. And that cross product we use also relates to the increment of area in integrals.

EXAMPLE 9 Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$, $z = u + 2v$ at the point $(1, 1, 3)$.

$u =$

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} = 2u\,\mathbf{i} + \mathbf{k} \quad = \langle 2u, 0, 1\rangle = \bar{r}_u$$

$$\mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k} = 2v\,\mathbf{j} + 2\,\mathbf{k} \quad = \langle 0, 2v, 2\rangle = \bar{r}_v$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = -2v\,\mathbf{i} - 4u\,\mathbf{j} + 4uv\,\mathbf{k} \quad = \langle -2v, -4u, 4uv\rangle = \bar{n}$$

to the plane!

$x = u^2$
$y = v^2$
$z = u + 2v$

Therefore an equation of the tangent plane at $(1, 1, 3)$ is

$$-2(x - 1) - 4(y - 1) + 4(z - 3) = 0$$

I'm fine w/

$\bar{r} = \langle u^2, v^2, u+2v\rangle$

$= \langle 1, 1, 3\rangle$: $u = v = 1$ does it

$\langle 1,1,3\rangle + s\langle 2,0,1\rangle + t\langle 0,2,2\rangle$

$\bar{r} = \bar{r}_0 + u\bar{a} + v\bar{b}$

where $\bar{a}$ & $\bar{b}$ are any (independent) vectors "in" the plane

$x = 1 + 2s$
$y = 1 + 2t$
$z = 3 + s + 2t$

$-2x + 2 - 4y + 4 + 4z - 12 =$

$-2x - 4y + 4z = 6$

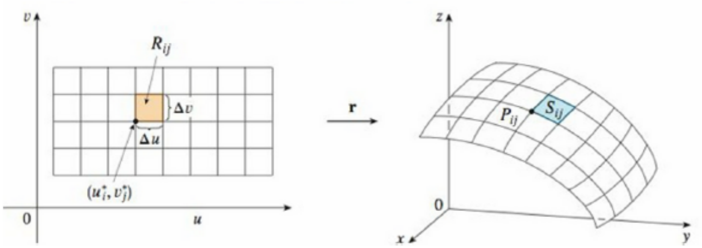
$-2(1) - 4(1) + 4(3) = 6$ ✓

GOOD CHECK Keenan.

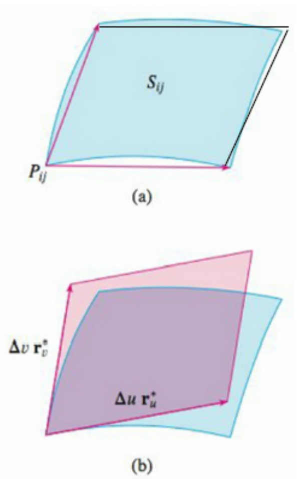
$\langle 2, 0, 1\rangle \; 2, 0$
$\langle 0, 2, 2\rangle \; 0, 2$

$\langle -2, -4, 4\rangle \leftrightarrow \langle 1, 2, -2\rangle$

$\langle -2, -4, 4\rangle \leftrightarrow \langle 1, 2, -2\rangle$



Nothing we do here is different than what we've done before. We're just applying more general techniques. The same approximation using partial derivatives and the area of a parallelogram via cross products....



6 DEFINITION If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D$$

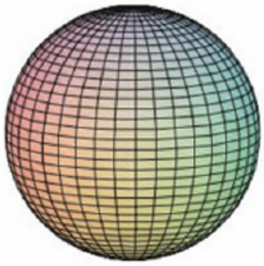
and S is covered just once as (u, v) ranges throughout the parameter domain D, then the **surface area** of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA$$

where $\quad \mathbf{r}_u = \dfrac{\partial x}{\partial u}\mathbf{i} + \dfrac{\partial y}{\partial u}\mathbf{j} + \dfrac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \dfrac{\partial x}{\partial v}\mathbf{i} + \dfrac{\partial y}{\partial v}\mathbf{j} + \dfrac{\partial z}{\partial v}\mathbf{k}$

This is one of the reasons I'd rather students remember the cross-product thing than necessarily memorize the Jacobian.

EXAMPLE 10 Find the surface area of a sphere of radius a.



SOLUTION In Example 4 we found the parametric representation

$$x = a \sin \phi \cos \theta \qquad y = a \sin \phi \sin \theta \qquad z = a \cos \phi$$

$$D = \{(\phi, \theta) \mid 0 \leqslant \phi \leqslant \pi,\ 0 \leqslant \theta \leqslant 2\pi\}$$

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial \phi} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial z}{\partial \phi} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= a^2 \sin^2\phi \cos\theta\, \mathbf{i} + a^2 \sin^2\phi \sin\theta\, \mathbf{j} + a^2 \sin\phi \cos\phi\, \mathbf{k}$$

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{a^4 \sin^4\phi \cos^2\theta + a^4 \sin^4\phi \sin^2\theta + a^4 \sin^2\phi \cos^2\phi}$$

$$= \sqrt{a^4 \sin^4\phi + a^4 \sin^2\phi \cos^2\phi} = a^2\sqrt{\sin^2\phi} = a^2 \sin\phi$$

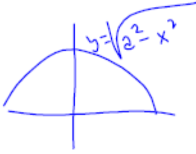
$\sin\phi \geqslant 0$

$$A = \iint_D |\mathbf{r}_\phi \times \mathbf{r}_\theta|\, dA = \int_0^{2\pi}\int_0^{\pi} a^2 \sin\phi\, d\phi\, d\theta$$

$$= a^2 \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\, d\phi = a^2(2\pi)2 = 4\pi a^2$$

$$\int_0^{\pi} \sin\phi\, d\theta = -\cos\phi \Big]_0^{\pi} = -(-1)-(-1) = 2$$

Calc I :



$$2\pi \int_a^b y \boxed{\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx} = 2\pi \int y\, ds$$

$$2\pi \int_{-a}^{a} \sqrt{a^2-x^2} \sqrt{1 + \frac{1}{4}\left(\frac{1}{a^2-x^2}\right)(-2x)}\, dx$$

$(a^2-x^2)^{\frac{1}{2}}$

SURFACE AREA OF THE GRAPH OF A FUNCTION

For the special case of a surface S with equation , where (x, y) lies in D and f has continuous partial derivatives, we take x and y as parameters. The parametric equations are

$z = f(x, y)$ $\qquad x = x \qquad y = y \qquad z = f(x, y)$

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}\right)\mathbf{k}$$

7
$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \dfrac{\partial f}{\partial x} \\ 0 & 1 & \dfrac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$$

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

This is all review, except we didn't write everything in vector notation.

We consider the surface S obtained by rotating the curve $y = f(x)$, $a \leqslant x \leqslant b$, about the x-axis, where $f(x) \geqslant 0$ and f' is continuous. From Equations 3 we know that parametric equations of S are

$$x = x \qquad y = f(x)\cos\theta \qquad z = f(x)\sin\theta \qquad a \leqslant x \leqslant b \qquad 0 \leqslant \theta \leqslant 2\pi$$

To compute the surface area of S we need the tangent vectors

$$\mathbf{r}_x = \mathbf{i} + f'(x)\cos\theta\,\mathbf{j} + f'(x)\sin\theta\,\mathbf{k}$$

$$\mathbf{r}_\theta = -f(x)\sin\theta\,\mathbf{j} + f(x)\cos\theta\,\mathbf{k}$$

Thus

$$\mathbf{r}_x \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(x)\cos\theta & f'(x)\sin\theta \\ 0 & -f(x)\sin\theta & f(x)\cos\theta \end{vmatrix}$$

$$= f(x)f'(x)\,\mathbf{i} - f(x)\cos\theta\,\mathbf{j} - f(x)\sin\theta\,\mathbf{k}$$

and so $$|\mathbf{r}_x \times \mathbf{r}_\theta| = \sqrt{[f(x)]^2[f'(x)]^2 + [f(x)]^2\cos^2\theta + [f(x)]^2\sin^2\theta}$$

$$= \sqrt{[f(x)]^2[1 + [f'(x)]^2]} = f(x)\sqrt{1 + [f'(x)]^2}$$

because $f(x) \geqslant 0$. Therefore the area of S is

$$A = \iint_D |\mathbf{r}_x \times \mathbf{r}_\theta|\,dA = \int_0^{2\pi}\int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx\,d\theta$$

$$= 2\pi\int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx$$

Recall Surface Area of a Surface of Revolution?

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx$$

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

This ties together the increment of arc length with what we're doing.

1–2 Determine whether the points P and Q lie on the given surface.

1. $\mathbf{r}(u, v) = \langle 2u + 3v, 1 + 5u - v, 2 + u + v \rangle$
$P(7, 10, 4)$, $Q(5, 22, 5)$

3–6 Identify the surface with the given vector equation.

$\boxed{\textbf{3.}}$ $\mathbf{r}(u, v) = (u + v)\,\mathbf{i} + (3 - v)\,\mathbf{j} + (1 + 4u + 5v)\,\mathbf{k}$ Dylan

5. $\mathbf{r}(s, t) = \langle s, t, t^2 - s^2 \rangle$

$z = y^2 - x^2$