

Suppose  $z = f(x, y)$  has continuous partials.

Then the **tangent plane** to  $f$  at the point  $P(x_0, y_0, z_0)$

is  $z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + z_0$

If  $z = f(x, y)$ , we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

**CLAIRAUT'S THEOREM** Suppose  $f$  is defined on a disk  $D$  that contains the point  $(a, b)$ . If the functions  $f_{xy}$  and  $f_{yx}$  are both continuous on  $D$ , then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

The total differential  $dz = f_x(x, y) dx + f_y(x, y) dy = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

**[2] THE CHAIN RULE (CASE 1)** Suppose that  $z = f(x, y)$  is a differentiable function of  $x$  and  $y$ , where  $x = g(t)$  and  $y = h(t)$  are both differentiable functions of  $t$ .

Then  $z$  is a differentiable function of  $t$  and  $\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$

**[3] THE CHAIN RULE (CASE 2)** Suppose that  $z = f(x, y)$  is a differentiable function of  $x$  and  $y$ , where  $x = g(s, t)$  and  $y = h(s, t)$  are differentiable functions of  $s$  and  $t$ . Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

$$F(x, y) = 0 \Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y} \quad F(x, y, z) = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

If  $f$  is a differentiable function of  $x$  and  $y$ , then  $f$  has a directional derivative in the direction of any unit vector  $\mathbf{u} = \langle a, b \rangle$  and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

**GRADIENT VECTOR**  $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$

**tangent plane to the level surface**  $F(x, y, z) = k$  at  $P(x_0, y_0, z_0)$

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

If  $f$  has a local maximum or minimum at  $(a, b)$  and the first-order partial derivatives of  $f$  exist there, then  $f_x(a, b) = 0$  and  $f_y(a, b) = 0$ .

$$f_x(a, b) = 0, f_y(a, b) = 0 \text{ and } D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 \Rightarrow$$

- (a) If  $D > 0$  and  $f_{xx}(a, b) > 0$ , then  $f(a, b)$  is a local minimum.
- (b) If  $D > 0$  and  $f_{xx}(a, b) < 0$ , then  $f(a, b)$  is a local maximum.
- (c) If  $D < 0$ , then  $f(a, b)$  is not a local maximum or minimum.

**METHOD OF LAGRANGE MULTIPLIERS** To find the maximum and minimum values of  $f(x, y, z)$  subject to the constraint  $g(x, y, z) = k$  [assuming that these extreme values exist and  $\nabla g \neq \mathbf{0}$  on the surface  $g(x, y, z) = k$ ]:

- (a) Find all values of  $x, y, z$ , and  $\lambda$  such that  $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$
- (b) Evaluate  $f$  at all the points  $(x, y, z)$  that result from step (a). The largest of these values is the maximum value of  $f$ ; the smallest is the minimum value of  $f$ .