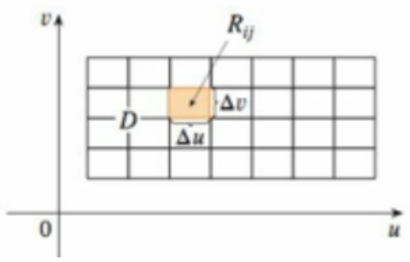
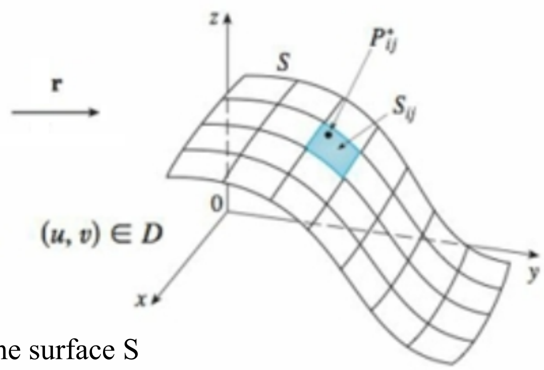
SURFACE INTEGRALS





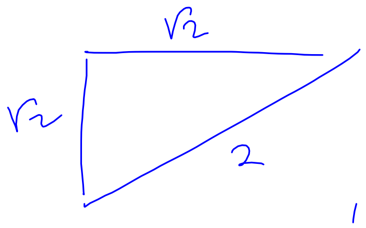
$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D$

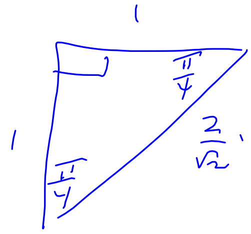
We want to integrate a function f over the surface S

$$\iint_S f(x, y, z)\, dS = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\, \Delta S_{ij}$$

dS is the increment of area on the surface S.

$\Delta S_{ij} \approx |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u\, \Delta v$ and we pass to the limit...





$\frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \sqrt{2}$

$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

This increment of area should be no surprise to you.

$$\iint_S 1\, dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA = A(S)$$ = surface area

EXAMPLE 1 Compute the surface integral $\iint_S x^2\, dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

Recall we computed the surface area of a sphere of radius a in the previous section.

While you may certainly put this special case on the cheat sheet, I sure wouldn't try to remember this, separately, other than the first little bit. If you get the first little bit, then you can derive the rest, which is generally better than memorizing a bunch of trivia.

GRAPHS

Any surface S with equation $z = g(x, y)$ can be regarded as a parametric surface with parametric equations

$\bar{r}(x,y) = \langle x, y, g(x,y) \rangle$

$$x = x \qquad y = y \qquad z = g(x, y)$$

and so we have

$$\boxed{\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial g}{\partial x}\right)\mathbf{k}} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial g}{\partial y}\right)\mathbf{k}$$

$\bar{r}_x = \langle 1, 0, \frac{\partial g}{\partial x} \rangle \qquad \bar{r}_y = \langle 0, 1, \frac{\partial g}{\partial y} \rangle$

Thus

3
$$\boxed{\mathbf{r}_x \times \mathbf{r}_y} = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}$$

and

$$|\mathbf{r}_x \times \mathbf{r}_y| = \boxed{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$$

Therefore, in this case, Formula 2 becomes

$\boxed{|\bar{r}_x \times \bar{r}_y|}$

4
$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

ORIENTED SURFACES

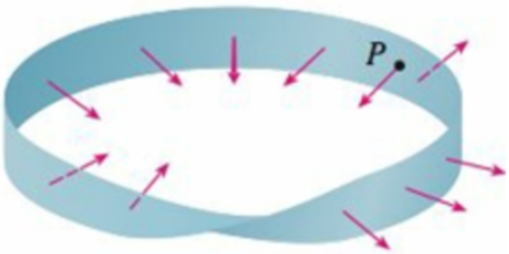
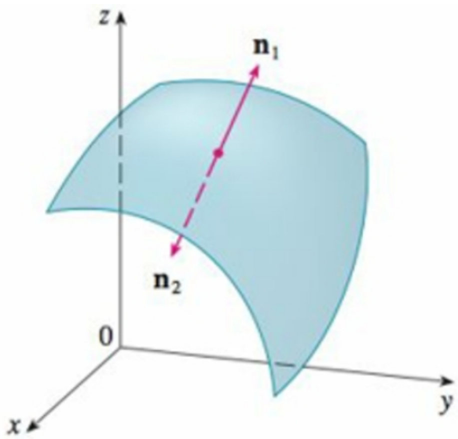


FIGURE 4
A Möbius strip

We convey the idea of an oriented surface by kicking things off with a NONEXAMPLE. The Mobius strip doesn't have an inside or outside. It's all just one side. Can't be oriented.

Klein Bottle



We only treat with orientable surfaces, that is, surfaces where you can pick an orientation, depending on which normal to the tangent plane at a point that you decide to choose.

Sometimes you'll use one. Sometimes the other. It just depends on the physical situation, or your preference. As long as you stick to the chosen orientation, either of the **n**'s is fine.

For a surface $z = g(x, y)$ given as the graph of g, we use Equation 3 to associate with the surface a natural orientation given by the unit normal vector

$$\boxed{5} \qquad \mathbf{n} = \frac{-\dfrac{\partial g}{\partial x}\mathbf{i} - \dfrac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial g}{\partial x}\right)^2 + \left(\dfrac{\partial g}{\partial y}\right)^2}} = \frac{\bar{r}_u \times \bar{r}_v}{\|\bar{r}_u \times r_v\|}$$

Since the **k**-component is positive, this gives the *upward* orientation of the surface.

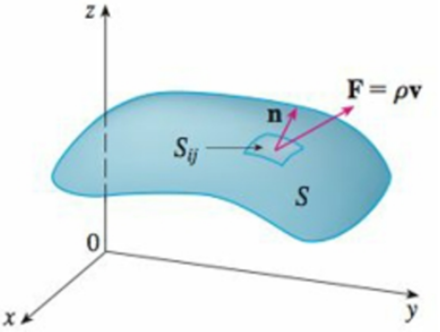
This just makes sure that the **n** points UP. More generally,

If S is a smooth orientable surface given in parametric form by a vector function $\mathbf{r}(u, v)$, then it is automatically supplied with the orientation of the unit normal vector

$$\boxed{6} \qquad \mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

This is standard orientation.

SURFACE INTEGRALS OF VECTOR FIELDS



$\rho = \frac{Kg}{m^3}$

$\bar{v} = \frac{m}{s}$

Suppose that S is an oriented surface with unit normal vector $\mathbf{n}$, and imagine a fluid with density $\rho(x, y, z)$ and velocity field $\mathbf{v}(x, y, z)$ flowing through S. (Think of S as an imaginary surface that doesn't impede the fluid flow, like a fishing net across a stream.) Then the rate of flow (mass per unit time) per unit area is $\rho\mathbf{v}$.

$\rho\bar{v} \quad \frac{Kg}{m^3} \cdot \frac{m}{s} = \frac{\frac{Kg}{m^2}}{s}$

The rate of flow through the patch S_{ij} is given by

$$(\rho\mathbf{v} \cdot \mathbf{n})A(S_{ij})$$

$\frac{\frac{Kg}{m}}{s} \cdot \frac{Kg}{m^2\text{-}s} \quad \frac{Kg}{s\text{-}m^2}$

The dot product captures the component of the flow that is normal to the surface S, which means that it's capturing the flow THROUGH the boundary, which is cool.

rate

the surface integral of the function $\rho\mathbf{v} \cdot \mathbf{n}$

$$\iint_S \rho\mathbf{v} \cdot \mathbf{n}\, dS = \iint_S \rho(x, y, z)\mathbf{v}(x, y, z) \cdot \mathbf{n}(x, y, z)\, dS$$

and this is interpreted physically as the rate of flow through S.

This is just a specific case of integrating the normal component of a vector field over a surface.

8 DEFINITION If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the **surface integral of $\mathbf{F}$ over S** is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS \qquad \text{or } \textit{flux integral}$$

This integral is also called the **flux** of $\mathbf{F}$ across S.

Notice the $d\mathbf{S}$ versus the dS ?

The first is a vector.

0

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\, dS$$

$$= \iint_D \left[\mathbf{F}(\mathbf{r}(u,v))\cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\right]|\mathbf{r}_u \times \mathbf{r}_v|\, dA = \iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

$d\bar{S} = \bar{n}\, dS$

I wouldn't sweat the following. If you understand the GENERAL, the following is what you get in the special case of a $z = g(x, y)$ situation, with parameters $x = x$ and $y = y$, as before.

In the case of a surface S given by a graph $z = g(x, y)$, we can think of x and y as parameters and use Equation 3 to write

$$\mathbf{F}\cdot(\mathbf{r}_x \times \mathbf{r}_y) = (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k})\cdot\left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}\right)$$

Thus Formula 9 becomes

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$$\boxed{\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA}$$

EXAMPLE 5 Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

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$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

Electrostatic field, **E**.

electric flux of **E** through the surface S.

Gauss's Law says that the net charge enclosed by a closed surface S is

$$Q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S}$$

where $\varepsilon_0 \approx 8.8542 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2}$ is the permittivity of free space

HEAT FLOW

Suppose the temperature at a point (x, y, z) in a body is $u(x, y, z)$.

Then the **heat flow** is defined as the vector field

$$\mathbf{F} = -K\,\nabla u$$

where K is an experimentally determined constant called the **conductivity** of the substance.

The rate of heat flow across the surface S in the body is then given by the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -K \iint_S \nabla u \cdot d\mathbf{S} \quad = -K \iint_{S'} \nabla u \cdot \bar{n}\, dS'$$

$$= -K \iint_{S'} \nabla u \cdot \bar{r}_u \times \bar{r}_v \, \underbrace{du\,dv}_{dA}$$

in the pre-image (Domain)