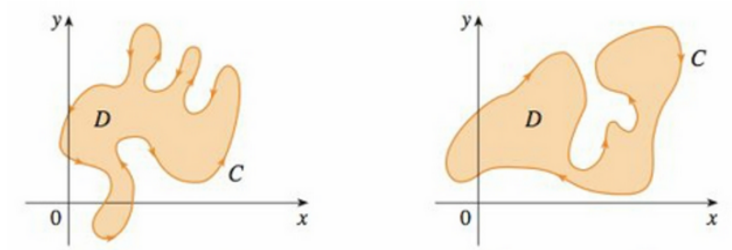
GREEN'S THEOREM



(a) Positive orientation (b) Negative orientation

GREEN'S THEOREM Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial D} P\,dx + Q\,dy$$

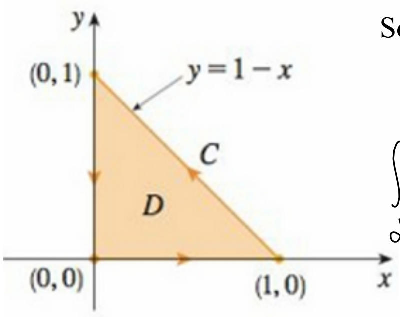
$$= \iint_D \operatorname{curl}(\bar{F}) \cdot \bar{k}\, dA \qquad = \int_{\partial D} \langle P(x(t),y(t)), Q(x(t),y(t))\rangle \cdot \langle x'(t), y'(t)\rangle\, dt$$

The notation $\oint_C P\,dx + Q\,dy$ or $\oint_C P\,dx + Q\,dy$ means positive orientation of the closed curve C.

$$= \int_{\partial D} \bar{F}(\bar{r}(t)) \cdot \bar{r}'(t)\, dt$$

Quick Application

EXAMPLE 1 Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from (0, 0) to (1, 0), from (1, 0) to (0, 1), and from (0, 1) to (0, 0).



Sometimes the double integral is easier.

$P_y = 0 \quad , \quad Q_x = y$

TI integral!

$$\iint_{D} (Q_x - P_y)\, dA = \iint_D y\, dA$$

$$= \int_0^1 \int_0^{1-x} y\, dy\, dx = \int_0^1 \left[\tfrac{1}{2} y^2\right]_0^{(1-x)} dx$$

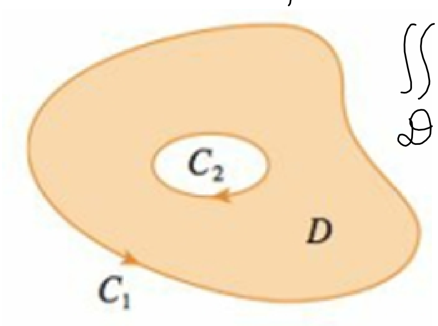
$$= \int_0^1 \tfrac{1}{2} (x-1)^2\, dx = \tfrac{1}{2}\left[\tfrac{1}{3}(x-1)^3\right]\Big]_0^1 = \tfrac{1}{6}\left[(0-1)^3\right]$$

$$= \boxed{-\tfrac{1}{6}}$$

Handling connected regions with holes.

Positive orientation means the interior is kept on the left as you traverse a closed curve. So positive orientation for C_2 is *clockwise*. But we still don't (maybe) see that the double integral over D is going to work, until we observe that it must agree with the sum of the integrals over D' and D''. And neither of the two sub-domains contains a hole.

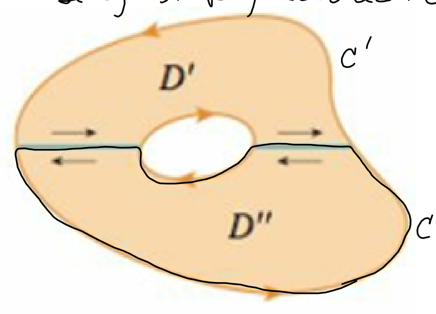
Not simply connected

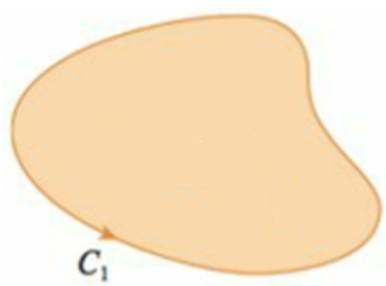


$$\iint_{D} = \iint_{D'} + \iint_{D''}$$

$$\int_{C_1} = \int_{C''} + \int_{C'}$$

Two domains, each of which being simply connected





It's interesting to see what happens when F has a discontinuity - some might call it a pole (and its graph DOES appear to have a giant pole at the origin, around which we draw our closed curve C.

In years to come: Cauchy Integral Theorem. Residue Theory. Complex Analysis.

Keep this example lodged in long-term memory. Line integrals are of major importance.

When integrating over huge swaths of the plane or in space, this theory tells us that it all amounts to nothing, except at the poles, where there's a residue that you don't get when integrating everywhere-smooth on closed curves. Poles only occur where the function F has a discontinuity. So you ferret out where it's NOT continuous, knowing there's nothing else going on anywhere else!

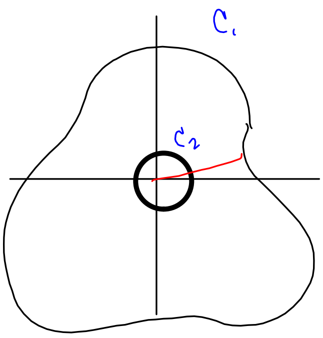
EXAMPLE 5 If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

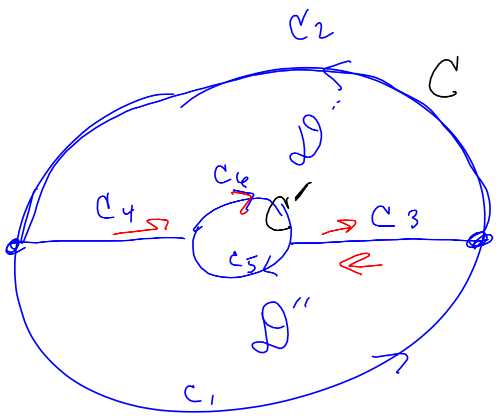
$$\mathbf{F} = (-y\mathbf{i} + x\mathbf{j})/(x^2+y^2) = \langle -y, x\rangle/(x^2+y^2)$$

$$= \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$ is how I'd represent it.

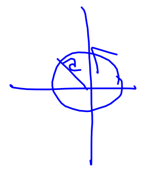
$$\int_{C_1} \bar{F} \cdot d\bar{r} = 2\pi$$

$$\int_{C_2} \bar{F} \cdot d\bar{r} = 2\pi$$





Keep Domain on your left.



$$\iint_D = \iint_{D'} + \iint_{D''}$$

$$= \int_{C_4} + \int_{C_6} + \int_{C_3} + \int_{C_2} + \int_{C_1} + \int_{-C_3} + \int_{C_5} + \int_{-C_4}$$

$$= \int_{C_1} + \int_{C_2} + \int_{C_6} + \int_{C_5}$$

$$= \int_C + \int_{-C'} = 0$$

$$\int_C = -\int_{-C'} = \int_{C'}$$

$$\iint_{\mathcal{D}} \underline{\left(\frac{y^2-x^2}{x^2+y^2} - \frac{y^2-x^2}{x^2+y^2} \right)} dA = 0$$

$$\bar{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle = \langle P, Q \rangle$$

$$\int \bar{F} \cdot d\bar{r} = \int \langle P, Q \rangle \cdot d\bar{r} = \iint (Q_x - P_y)\, dy$$

$$\int_{C'} \bar{F} \cdot d\bar{r} =$$

C = circle of sufficiently small radius a.

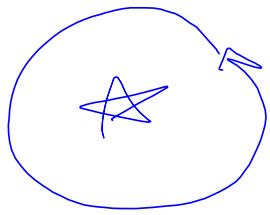
$$\bar{r} = \langle a\cos\theta, a\sin\theta \rangle$$

$$d\bar{r} = \langle -a\sin\theta, a\cos\theta \rangle\, d\theta$$

$$\frac{-y}{x^2+y^2} = \frac{-a\sin\theta}{a^2}$$

$$\int_{C'} \bar{F} \cdot d\bar{r} = \int_0^{2\pi} \left\langle -\frac{\sin\theta}{a}, \frac{\cos\theta}{a} \right\rangle \cdot \langle -a\sin\theta, a\cos\theta \rangle\, d\theta$$

$$= \int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\, d\theta = \int_0^{2\pi} d\theta = \theta \Big]_0^{2\pi} = 2\pi$$



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