# THE FUNDAMENTAL THEOREM FOR LINE INTEGRALS

Recall, FTC II: $$\int_a^b F'(x)\,dx = F(b) - F(a)$$

**2 THEOREM** Let $C$ be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Let $f$ be a differentiable function of two or three variables whose gradient vector $\nabla f$ is continuous on $C$. Then


$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \qquad \nabla f = \langle x_t, y_t, z_t \rangle$$

Basically, line integrals of conservative vector fields are independent of path.

And if it's independent of path, and the path is closed (ends where it started), then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

**3 THEOREM** $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in $D$ if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path $C$ in $D$.

Conservative Vector Field $= \bar{F}(x,y) = \langle P, Q \rangle$

means $\bar{F} = \nabla f$ for some function $f(x,y)$

$$f(x,y) = x^2y + \cos x \cos y$$

$$f_x = 2xy - \sin x \cos y$$

$$f_y = x^2 - \cos x \sin y$$

$$\bar{F} = \langle 2xy - \sin x \cos y,\ x^2 \cos x \sin y \rangle$$

$$\nabla = \left\langle \frac{d}{dx}, \frac{d}{dy} \right\rangle$$

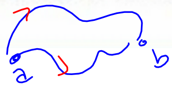
$$\nabla f = \left\langle \frac{d}{dx}, \frac{d}{dy} \right\rangle f = \left\langle \frac{df}{dx}, \frac{df}{dy} \right\rangle$$

$\nabla \bar{F}$ doesn't exist, but

$\nabla \cdot \bar{F}$ is used instead.

$$= \langle P_x, Q_y \rangle = \left\langle \frac{d}{dx}, \frac{d}{dy} \right\rangle \cdot \langle P, Q \rangle$$

**4 THEOREM** Suppose **F** is a vector field that is continuous on an open connected region $D$. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in $D$, then **F** is a conservative vector field on $D$; that is, there exists a function $f$ such that $\nabla f = \mathbf{F}$.




So, F is conservative if and only if the corresponding line integrals are independent of path.

You want to show it's independent of path? Show it's conservative.

You want to show it's conservative? Prove it's independent of path (not always easy).

$F = \langle P, Q \rangle$

**5 THEOREM** If $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is a conservative vector field, where $P$ and $Q$ have continuous first-order partial derivatives on a domain $D$, then throughout $D$ we have

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

This is just Clairaut's Theorem.

If F is a gradient for some f, then this equation always holds.

$F(x,y) = \langle P(x,y), Q(x,y) \rangle$

$F$ conservative $\Rightarrow F = \nabla f$

$F = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle P, Q \rangle$

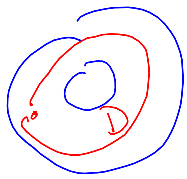
$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$

A **simple curve** is a curve that doesn't intersect itself anywhere between its endpoints.

$F$

A **simply-connected region** in the plane is a connected region $D$ such that every simple closed curve in $D$ encloses only points that are in $D$. $\Leftrightarrow$ No holes in D.

This is a fancy mathematical way of saying that $D$ has no holes!




**6 THEOREM** Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region $D$. Suppose that $P$ and $Q$ have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

$\langle f_{xy}, f_{yx} \rangle$

Then **F** is conservative.

$F = \langle P, Q \rangle = \langle f_x, f_y \rangle$ for some $f$.

For MOST intents and purposes, Theorem 6 gives us a working converse to Theorem 5. You just want to be careful not to think everything's hunky-dory, when the domain has a hole, or the field is undefined somewhere inside the domain. All this theory requires the field to be nice and continuous (which implies bounded). $\Rightarrow$ smooth

## CONSERVATION OF ENERGY

$\vec{F} = ma$

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$$

So the work done by the force on the object is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_a^b m\mathbf{r}''(t) \cdot \mathbf{r}'(t)\,dt$$

$$= \frac{m}{2}\int_a^b \frac{d}{dt}[\mathbf{r}'(t) \cdot \mathbf{r}'(t)]\,dt \qquad \text{(Theorem 14.2.3, Formula 4)}$$

$$= \frac{m}{2}\int_a^b \frac{d}{dt}|\mathbf{r}'(t)|^2\,dt = \frac{m}{2}\left[|\mathbf{r}'(t)|^2\right]_a^b \qquad \text{(Fundamental Theorem of Calculus)}$$

$$= \frac{m}{2}\left(|\mathbf{r}'(b)|^2 - |\mathbf{r}'(a)|^2\right)$$

$$W = \tfrac{1}{2}m|\mathbf{v}(b)|^2 - \tfrac{1}{2}m|\mathbf{v}(a)|^2$$

Product Rule for Dot Product is just like ordinary product

$[fg]' = f'g + fg'$

$[\bar{r}' \cdot \bar{r}']' = \bar{r}'' \cdot \bar{r}' + \bar{r}' \cdot \bar{r}''$

$= \bar{r}'' \cdot \bar{r}' + \bar{r}'' \cdot \bar{r}'$

$= 2\bar{r}'' \cdot \bar{r}'$

$\Rightarrow \bar{r}'' \cdot \bar{r}' = \frac{1}{2}[\bar{r}' \cdot \bar{r}']'$

$K(t) = \frac{1}{2}m|\mathbf{v}(t)|^2 =$ **kinetic energy** of the object.

$\frac{1}{2}mv^2 =$ Kinetic Energy!

$W = K(B) - K(A)$ Work = the net change in kinetic energy.

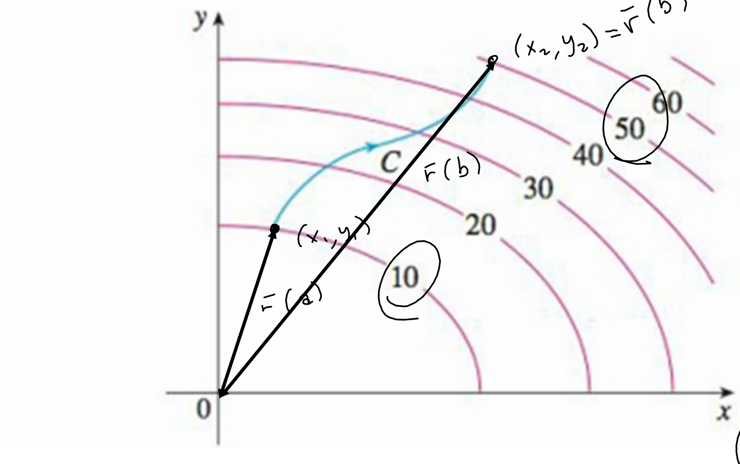
**potential energy** $= P(x, y, z) = -f(x, y, z)$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C \nabla P \cdot d\mathbf{r} = -[P(\mathbf{r}(b)) - P(\mathbf{r}(a))] = P(A) - P(B)$$

$\bar{F} = \nabla f$

$\int_C \nabla f \cdot d\bar{r}$

**1.** The figure shows a curve $C$ and a contour map of a function $f$ whose gradient is continuous. Find $\int_C \nabla f \cdot d\mathbf{r}$.




Fundamental Theorem

$$\int_C \nabla f \cdot d\bar{r} = f(\bar{r}(b)) - f(\bar{r}(a))$$

$$= \int_a^b \nabla f(\bar{r}(t)) \cdot \bar{r}'(t)\, dt$$

$$= 50 - 10 = 40!$$

$$= f(\bar{r}(b)) - f(\bar{r}(a))$$

$$\int \bar{F} \cdot d\bar{r} = \int \nabla f \cdot d\bar{r}$$

& everything's nice, for Conservative Vector Field $F$

**3–10** Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function $f$ such that $\mathbf{F} = \nabla f$.

**7.** $\mathbf{F}(x, y) = (ye^x + \sin y)\,\mathbf{i} + (e^x + x\cos y)\,\mathbf{j}$

By Thm 6: $\langle ye^x + \sin y,\ e^x + x\cos y \rangle$

$P_y = e^x + \cos y$

$Q_x = e^x + \cos y$

$\Longrightarrow \bar{F}$ is conservative, in the sequel.

Variation of Parameters

$g(y, z)$

$f = \int f_x\, dx + g(y) = ye^x + x\sin y + g(y)$

$\frac{d}{dy}(\sin y)$ $\frac{d}{dy}(\sin y) = \cos y$

$f_y = e^x + x\cos y + g'(y) = Q = e^x + x\cos(y) \Longrightarrow$

$g'(y) \equiv 0$

$\Longrightarrow g(y) = C$

$f(x,y) = \boxed{ye^x + x\sin(y)} + C \quad \forall\, c \in \mathbb{R}$

$\int_C \bar{F} \cdot d\bar{r} = f(\bar{r}(b)) - f(\bar{r}(a))$

we don't know where C starts or ends.

when $g(y)$ ain't trivial

$f(x,y) = x^2 y + y^3$ → The $g(y)$ is $y^3$.

$f_x = 2xy$

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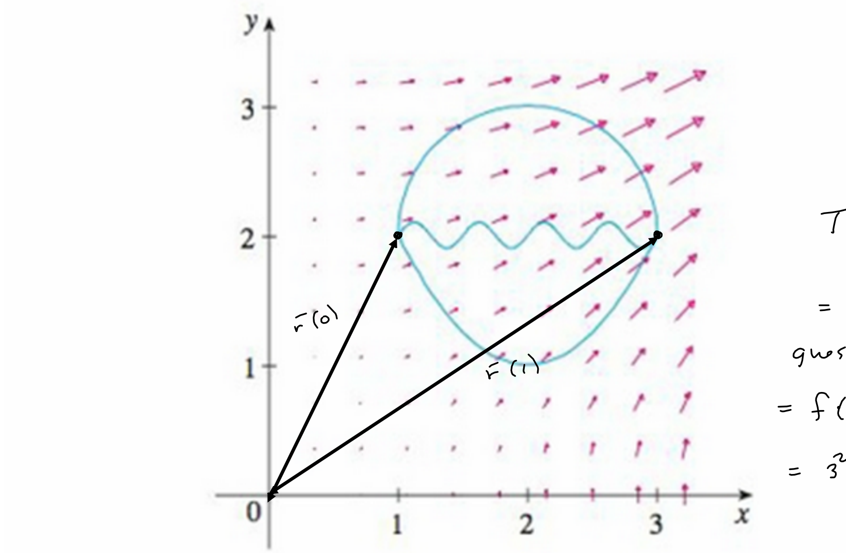
**11.** The figure shows the vector field $\mathbf{F}(x, y) = \langle 2xy, x^2 \rangle$ and three curves that start at (1, 2) and end at (3, 2).

(a) Explain why $\int_C \mathbf{F} \cdot d\mathbf{r}$ has the same value for all three curves.

(b) What is this common value?




$P_y = 2x = Q_x \Longrightarrow$ conservative.

$f_x = 2xy$

$\Longrightarrow f = x^2 y + g(y)$

$f_y = x^2 \Longrightarrow$

$f = x^2 y + g(x)$

$g(x) = g(y) \Longrightarrow g(x) = C = g(y)$

Choose $C = 0$

Then $\int_C \vec{F} \cdot d\vec{r}$

$= f(\vec{r}(1)) - f(\vec{r}(0))$

guesstimate from the field

$= f(3,2) - f(1,2)$

$= 3^2 \cdot 2 - 1^2 \cdot 2 = 8 \cdot 2 = \boxed{16}$

**12–18** (a) Find a function $f$ such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve $C$.

**12.** $\mathbf{F}(x, y) = x^2\,\mathbf{i} + y^2\,\mathbf{j}$,
$C$ is the arc of the parabola $y = 2x^2$ from $(-1, 2)$ to $(2, 8)$

**13.** $\mathbf{F}(x, y) = xy^2\,\mathbf{i} + x^2y\,\mathbf{j}$,
$C$: $\mathbf{r}(t) = \langle t + \sin\frac{1}{2}\pi t,\ t + \cos\frac{1}{2}\pi t \rangle$, $\quad 0 \leqslant t \leqslant 1$

$\bar{F} = \nabla f = \langle x^2, y^2 \rangle = \langle f_x, f_y \rangle$

$f_x = x^2 \longrightarrow$

$f = \int x^2\,dx + g(y) = \frac{1}{3}x^3 + g(y)$

$f_y = 0 + g'(y) = y^2$

$g(y) = \frac{1}{3}y^3 + C$

$\longrightarrow \boxed{f = \frac{1}{3}x^3 + \frac{1}{3}y^3} + \underbrace{\text{Any}}\ \text{constant}$

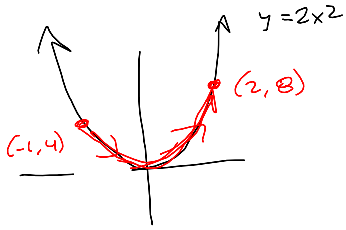
0 is nice.

$f = \int f_y\,dy + h(x)$

$= \int y^2\,dy + h(x)$

$= \frac{1}{3}y^3 + h(x) \overset{\text{SET}}{=} \frac{1}{3}x^3 + g(y)$

$\longrightarrow h(x) = \frac{1}{3}x^3$ & $g(y) = \frac{1}{3}y^3$




(b) $\int_C \bar{F} \cdot d\bar{r} = \int_C \nabla f \cdot d\bar{r}$

2!

You do it with (-1, 2)...

$= f(\bar{r}(2,8)) - f(\bar{r}(-1,4))$

$= \frac{1}{3}\left[(2^3 + 8^3) - ((-1)^3 + 4^3)\right]$

$= \frac{1}{3}\left[2^3 + 8^3 + 1^3 - 4^3\right]$

$= \frac{1}{3}\left[2^3 + 2^9 + 1 - 2^6\right]$

$= \frac{1}{3}\left[2^3(1 + 2^6 - 2^3) + 1\right] = \frac{1}{3}\left[8(57) + 1\right]$

$= \frac{1}{3}\left[456 + 1\right] = \frac{1}{3}\left[457\right] = \frac{457}{3}$

**19–20** Show that the line integral is independent of path and evaluate the integral.

$\frac{\partial}{\partial y}[\tan(y)] = \sec^2(y)$ ✓

**19.** $\int_C \tan y\,dx + x\sec^2 y\,dy$, $\frac{d}{dx}[x\sec^2(y)] = \sec^2(y)$

$C$ is any path from $(1, 0)$ to $(2, \pi/4)$

any path ? $\bar{r}(t) = (1-t)\langle 1,0\rangle + t\langle 2, \frac{\pi}{4}\rangle$

$\bar{F}(x(t),y(t)) = \langle \tan(y(t)), x(t)\sec^2(y(t))\rangle = \nabla f$ for some $f$

$f_x = \tan(y)$

$f = \int \tan(y)\,dx + g(y) = x\tan(y) + g(y)$

$f = \int x\sec^2(x)\,dy + g(x) = xy\sec^2(x) + g(x)$

$f_y = x\sec^2(y) + g'(y) = x\sec^2(y) \Rightarrow g'(y) = 0 \Rightarrow$

$g(y) = \text{Constant}$

$f = x\tan(y)$

$f(2,\frac{\pi}{4}) - f(1,0) = 2\tan\frac{\pi}{4} - 1\tan(0) = \boxed{2}$

$d\bar{r} = \langle x'(t), y'(t)\rangle\,dt$

$\bar{F}\cdot d\bar{r} = \langle \tan(y(t)), x(t)\sec^2(y(t))\rangle \cdot \langle x'(t), y'(t)\rangle\,dt$

$= \tan(y(t))\,x'(t)\,dt + x(t)\sec^2(y(t))\,\underline{y'(t)\,dt}$

$= \tan(y)\,dx + x\sec^2(y)\,dy$

**21–22** Find the work done by the force field **F** in moving an object from $P$ to $Q$.

**21.** $\mathbf{F}(x, y) = 2y^{3/2}\,\mathbf{i} + 3x\sqrt{y}\,\mathbf{j}$; $P(1, 1)$, $Q(2, 4)$

$\int_C \vec{F} \cdot d\vec{r}$

$\vec{F}$ conservative?

$P_y = 3y^{\frac{1}{2}} \overset{?}{=} Q_x = 3\sqrt{y} = 3y^{\frac{1}{2}}$ Yes.

Use the theorem.

$f_x = 2y^{\frac{3}{2}} \Rightarrow f = 2xy^{\frac{3}{2}} + g(y)$

$\Rightarrow f_y = 3xy^{\frac{1}{2}} + g'(y) = 3x\sqrt{y} \Rightarrow$

$0 \overset{\text{SET}}{=} g(y) = C \in \mathbb{R} \Rightarrow$

$\boxed{f(x,y) = 2xy^{3/2}}$

$f(2,4) - f(1,1) = 2(2)(4)^{\frac{3}{2}} - 2(1)(1)^{\frac{3}{2}}$

$= 2\cdot 2\cdot 2^3 - 2 = 32 - 2 = \boxed{30 = \text{WORK}}$

$\vec{r} = (1-t)\langle 1,1\rangle + t\langle 2,4\rangle$

$= \langle 1-t, 1-t\rangle + \langle 2t, 4t\rangle$

$= \langle 1+t, 1+3t\rangle$

$= \langle t+1, 3t+1\rangle$