16.9 Change of Variable(s) in Multiple Integrals

Recall the Substitution Rule: THIS FIRST PAGE OF NOTES IS MESSED-UP. DON'T KNOW WHAT I WAS THINKING. WILL BE RE-WORKED, MONDAY...

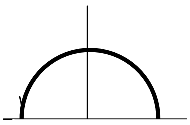
$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du$$

$x = g(u)$ and $a = g(c)$, $b = g(d)$

$$\int_a^b f(x)\,dx = \int_c^d f(x(u))\,\frac{dx}{du}\,du$$

Example: Trigonometric Substitution to simplify the integration process.

Find the area inside half a circle of radius $r = 1$.



We've already extended this to double integrals and polar coordinates:

$x = r\cos\theta$

$y = r\sin\theta$

$$\iint_R f(x, y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

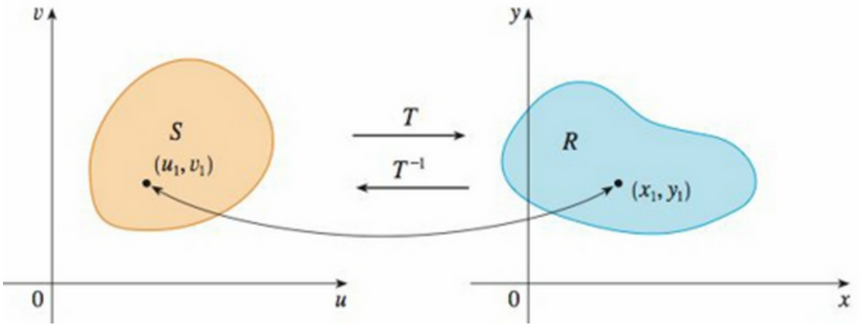
$$T(u, v) = (x, y)$$

$x = g(u, v)$ $\qquad$ $y = h(u, v)$

We usually assume that T is a C^1 **transformation**, which means that g and h have continuous first-order partial derivatives.

Transformations (Mappings) from one domain *onto* another.

If $T(u_1, v_1) = (x_1, y_1)$, then the point (x_1, y_1) is called the **image** of the point (u_1, v_1)



$$u = G(x, y) \qquad\qquad v = H(x, y)$$

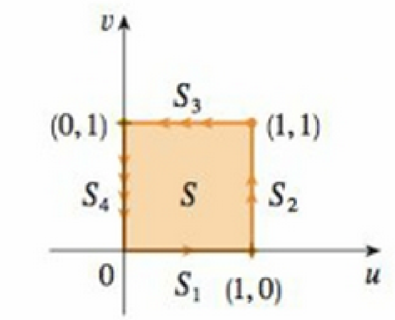
EXAMPLE 1 A transformation is defined by the equations

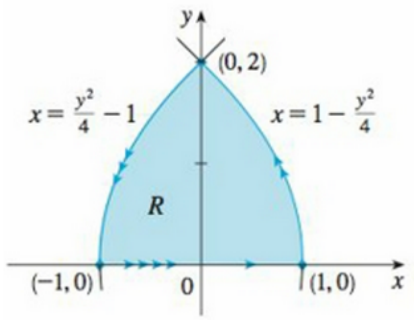
$$x = u^2 - v^2 \qquad y = 2uv$$

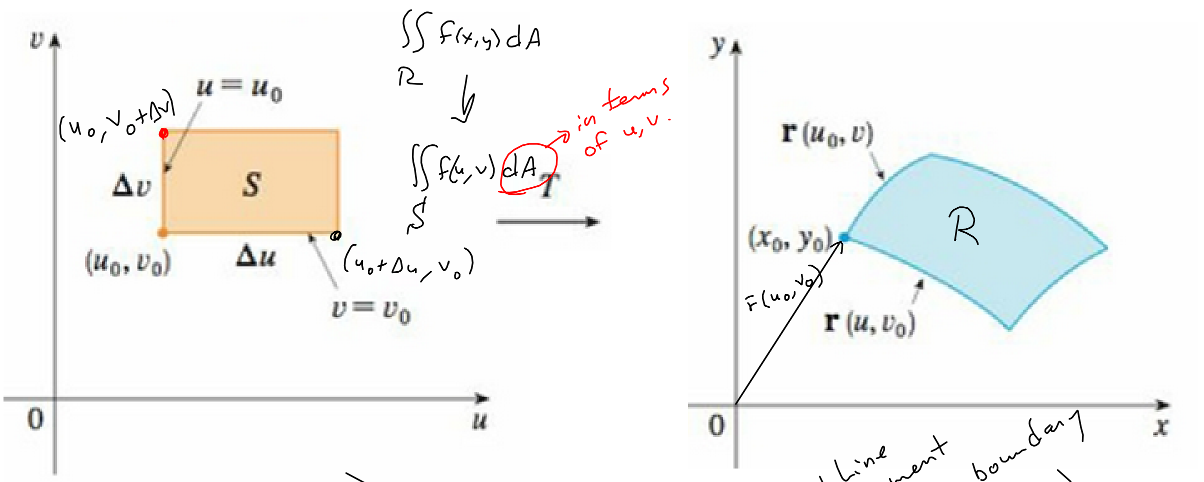
Find the image of the square $S = \{(u, v) \mid 0 \leqslant u \leqslant 1,\ 0 \leqslant v \leqslant 1\}$

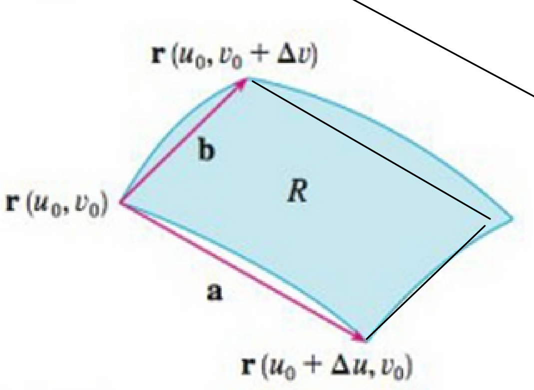
SOLUTION The transformation maps the boundary of S into the boundary of the image.

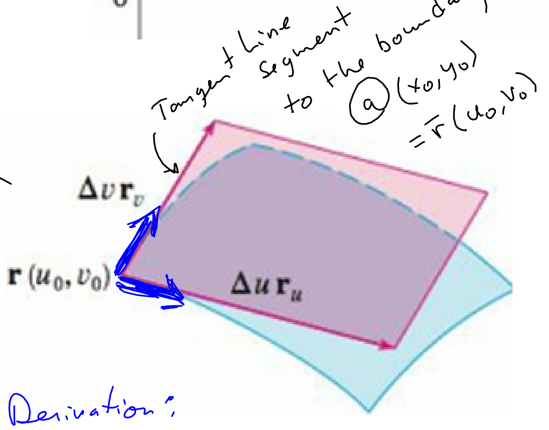
is reported as a fact, with no real proof, even though it is true.











Area of parallelogram is $\|\bar a \times \bar b\|$

Derivation:

$$\Delta \bar r = \bar b = \bar r(u_0, v_0+\Delta v) - \bar r(u_0, v_0)$$

$$\Rightarrow \frac{\Delta \bar r}{\Delta v} = \frac{\bar r(u_0, v_0+\Delta v) - \bar r(u_0, v_0)}{\Delta v} \approx \frac{\partial \bar r}{\partial v} = \bar r_v$$

$$\Rightarrow \bar b = \bar r(u_0, v_0+\Delta v) - \bar r(u_0, v_0)$$

$$= \frac{\bar r(u_0, v_0+\Delta v) - \bar r(u_0, v_0)}{\Delta v} \cdot \Delta v$$

$$\approx \frac{\partial \bar r}{\partial v} \cdot \Delta v = \bar r_v \Delta v$$

$$\bar a = \frac{\partial \bar r}{\partial u} \cdot \Delta u = \bar r_u \Delta u$$

$$\Longrightarrow \|\bar a \times \bar b\| \approx \|(\bar r_u \Delta u) \times (\bar r_v \Delta v)\| = \|\bar r_u \times \bar r_v\| \Delta u \Delta v = dA$$

The increment of area for the conversion!

$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Area of the region R using mapping T & domain S!

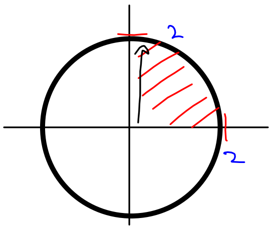
7 DEFINITION The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Double Integral in Polar Coordinates. Where did the $r\,dr\,d\theta$ come from?

We know, but here we do it in terms of the new machinery:

$$\int_0^1\int_0^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx$$



$x = r\cos\theta$, $y = r\sin\theta$

$T(r\cos\theta, r\sin\theta) = (x, y)$

$\bar{r}(r,\theta) = \langle r\cos\theta, r\sin\theta\rangle$

$\bar{r}_r = \langle \cos\theta, \sin\theta\rangle$

$\bar{r}_\theta = \langle -r\sin\theta, r\cos\theta$

$\|\bar{r}_r \times \bar{r}_\theta\|$

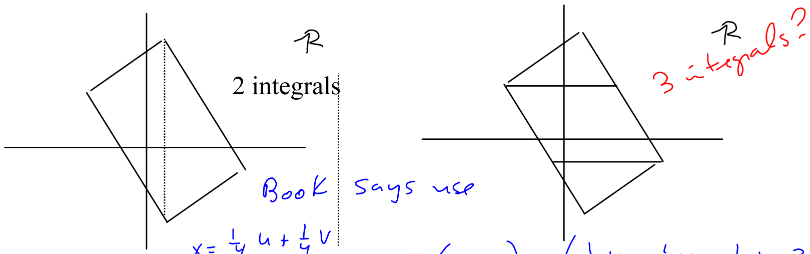
$\begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = |r\cos^2\theta + r\sin^2\theta|$

$= |r| = r$ if $0 \le r$

It's the r in the $r\,dr\,d\theta$!

#12

$\iint_R (4x+8y)\,dA$, where R is the parallelogram with ~~corners~~ corners $(-1,3)$, $(1,-3)$, $(3,-1)$, $(1,5)$



Book says use

$x = \frac{1}{4}u + \frac{1}{4}v$
$y = \frac{1}{4}v - \frac{3}{4}u$

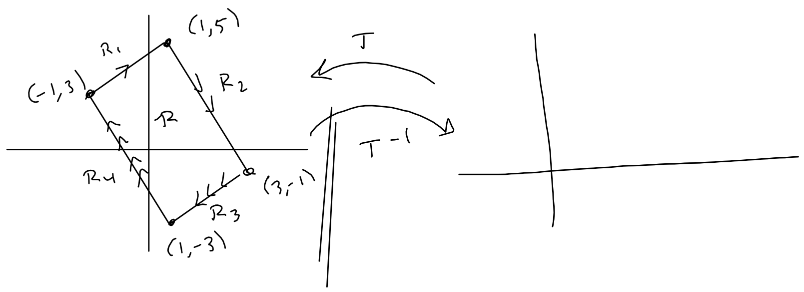
$T(u,v) = \left(\frac{1}{4}u + \frac{1}{4}v,\ \frac{1}{4}v - \frac{3}{4}u\right) = (x,y)$

$\vec{r}(u,v) = \left\langle \frac{1}{4}u + \frac{1}{4}v,\ \frac{1}{4}v - \frac{3}{4}u \right\rangle$

$\vec{r}_u = \left\langle \frac{1}{4}, -\frac{3}{4} \right\rangle,\quad \vec{r}_v = \left\langle \frac{1}{4}, \frac{1}{4} \right\rangle$

$$\begin{vmatrix} \frac{1}{4} & -\frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} \end{vmatrix} = \frac{1}{16} + \frac{3}{16} = \frac{4}{16} = \boxed{\frac{1}{4} = \frac{\partial(x,y)}{\partial(u,v)}}$$

$$4x + 8y = 4\left(\frac{1}{4}u + \frac{1}{4}v\right) + 8\left(\frac{1}{4}v - \frac{3}{4}u\right)$$
$$= u + v + 2v - 6u = -5u + 3v$$



To get T^{-1}, solve for u and v:

$x = \frac{1}{4}u + \frac{1}{4}v$
$y = \frac{1}{4}v - \frac{3}{4}u$

$$\begin{aligned} \tfrac{1}{4}u + \tfrac{1}{4}v &= x \\ -\tfrac{3}{4}u + \tfrac{1}{4}v &= y \end{aligned} \longrightarrow \left[\begin{array}{cc|c} \frac{1}{4} & \frac{1}{4} & x \\ -\frac{3}{4} & \frac{1}{4} & y \end{array}\right]$$

Reduced Row-Echelon Form: $\sim \left[\begin{array}{cc|c} 1 & 0 & x-y \\ 0 & 1 & 3x+y \end{array}\right]$

$T($

T^{-1}! $\Leftarrow$ $u = x - y$, $v = 3x + y$

$(-1,3)$, $(1,-3)$, $(3,-1)$, $(1,5)$

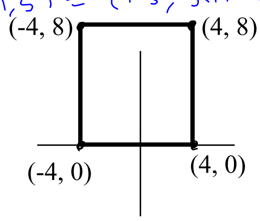
$T^{-1}(x,y) = (u,v) = (x-y,\ 3x+y)$

$T^{-1}(-1,3) = (-1-3,\ 3(-1)+3) = (-4,0)$

$T^{-1}(1,-3) = (1+3,\ 3(1)-3) = (4,0)$

$T^{-1}(3,-1) = \boxed{(3-1,\ 3(3)+1) = (2,10)}$ No. $(3-(-1),\ 3(3)+(-1)) = (4, 8)$

$T^{-1}(1,5) = (1-5,\ 3(1)+5) = (-4,8)$



1–6 Find the Jacobian of the transformation.

1. $x = 5u - v, \quad y = u + 3v$ **2.** $x = uv, \quad y = u/v$

3. $x = e^{-r}\sin\theta, \quad y = e^{r}\cos\theta$ **4.** $x = e^{s+t}, \quad y = e^{s-t}$

2 $x = uv$ $y = \frac{u}{v} = uv^{-1}$

$x_u = v$ $y_u = v^{-1} = \frac{1}{v}$

$x_v = u$ $y_v = -uv^{-2} = -\frac{u}{v^2}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} v & u \\ v^{-1} & -uv^{-2} \end{vmatrix} = \left| -uv^{-1} - uv^{-1} \right|$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \boxed{2\frac{u}{v} = \frac{2u}{v} = 2uv^{-1}}$$

7–10 Find the image of the set S under the given transformation.

7. $S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\}$;
$x = 2u + 3v,\ y = u - v$

8. S is the square bounded by the lines $u = 0$, $u = 1$, $v = 0$, $v = 1$; $\quad x = v,\ y = u(1 + v^2)$

11–16 Use the given transformation to evaluate the integral.

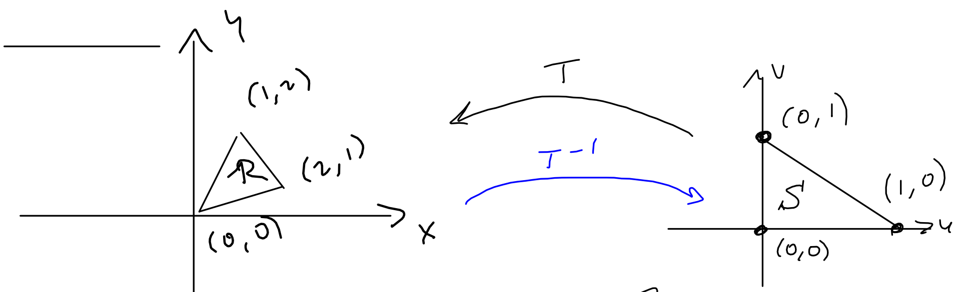
11. $\iint_R (x - 3y)\, dA$, where R is the triangular region with vertices (0, 0), (2, 1), and (1, 2); $x = 2u + v$, $y = u + 2v$

Defin: A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ satisfies

$$T(a\bar{u} + b\bar{v}) = aT(\bar{u}) + bT(\bar{v})$$

FACT: Any linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ can be represented by a matrix-times-vector product, i.e., T is a matrix!

Define $T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2u + 1v \\ 1u + 2v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$



$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $b = 1$, $d = 2$ $T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $a = 2$, $c = 1$

So $T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$x = 2u+v$ $\quad$ $y = u+2v$ $\quad$ $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$

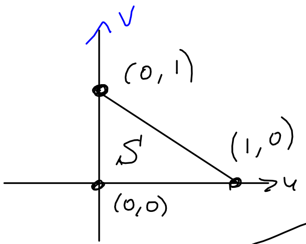
$x_u = 2$ $\quad$ $y_u = 1$

$x_v = 1$ $\quad$ $y_v = 2$

$$\iint_R (x-3y)\,dA = \iint_{S'} \big(2u+v-3(u+2v)\big)(3\,du\,dv)$$

$2u+v-3u-6v = -u-5v$

$$= -3\iint_{S'} (u+5v)\,du\,dv$$



$$= -3\int_0^1\int_0^{1-u} (u+5v)\,dv\,du$$

$$= -3\int_0^1\int_0^{1-v} (u+5v)\,du\,dv$$

$v = 1-u \Rightarrow u = 1-v$

$$= -3\int_0^1 \left[uv + \tfrac{5}{2}v^2\right]_0^{1-u} du = -3\int_0^1 \left(u(1-u) + \tfrac{5}{2}(1-u)^2\right) du$$

$u - u^2 + \frac{5}{2}(u^2 - 2u + 1) = u - u^2 + \frac{5}{2}u^2 - 5u + \frac{5}{2}$

$= \frac{3}{2}u^2 - 4u + \frac{5}{2}$

$$= -3\int_0^1 \left(\tfrac{3}{2}u^2 - 4u + \tfrac{5}{2}\right) du = -3\left[\tfrac{1}{2}u^3 - 2u^2 + \tfrac{5}{2}u\right]_0^1$$

$$-3\left[\tfrac{1}{2} - 2 + \tfrac{5}{2}\right] = \boxed{-3}$$

19–23 Evaluate the integral by making an appropriate change of variables.

19. $\iint_R \frac{x-2y}{3x-y}\,dA$, where R is the parallelogram enclosed by the lines $x-2y=0$, $x-2y=4$, $3x-y=1$, and $3x-y=8$

20. $\iint_R (x+y)e^{x^2-y^2}\,dA$, where R is the rectangle enclosed by the lines $x-y=0$, $x-y=2$, $x+y=0$, and $x+y=3$

19

$x-2y=0$ $\quad y=\frac{1}{2}x$

x	y
0	0

$x-2y=4$

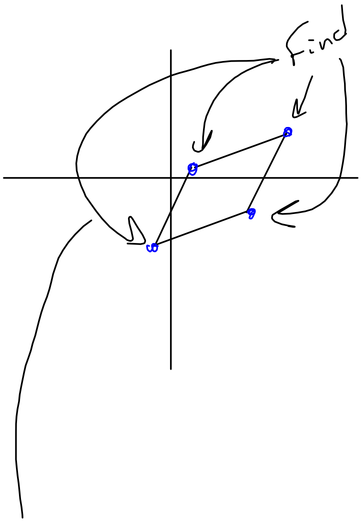
x	y
0	-2
4	0

$3x-y=1$

x	y
0	-1
$\frac{1}{3}$	0

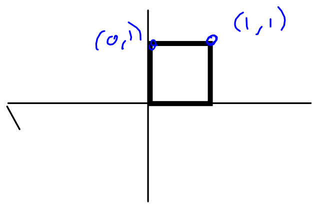
$3x-y=8$

x	y
0	-8
$\frac{8}{3}$	0



2 approaches:

1 Make the parallelogram a square.



2 Make a nice function

$$\frac{x-2y}{3x-y} \longrightarrow \frac{u}{v}$$

This requires finding $x(u,v)$ & $y(u,v)$

$u = x - 2y$

$v = 3x - y$

$T^{-1}: \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$

Want $(x, y) = T(u, v)$

Invert the matrix!

$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{array}\right] \xrightarrow[-3r_1+r_2]{r_1} \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 5 & -3 & 1 \end{array}\right]$

... So $T = \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -1 & 2 \\ -3 & 1 \end{bmatrix}$

Cramer's Rule

$\frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

~~$T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$~~

~~$T^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$~~

This says that

$x = -\frac{1}{5}u + \frac{2}{5}v$

$y = -\frac{3}{5}u + \frac{1}{5}v$

$\begin{vmatrix} -\frac{1}{5} & \frac{2}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{vmatrix} = -\frac{1}{25} + \frac{6}{25} = \frac{5}{25} = \frac{1}{5}$

$\iint_S \frac{u}{v} \cdot \frac{1}{5} \, du \, dv$

$\begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}$ will tell you the corner points of S

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