16.9 Change of Variable(s) in Multiple Integrals

Recall the Substitution Rule:

THIS FIRST PAGE OF NOTES IS MESSED-UP. DON'T KNOW WHAT I WAS THINKING. WILL BE RE-WORKED, MONDAY...

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du$$

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{u=g(a)}^{u=g(b)} f(u)\,du$$

$x = g(u)$ and $a = g(c)$, $b = g(d)$

$$\int_a^b f(x)\,dx = \int_c^d f(x(u))\frac{dx}{du}\,du$$

$x = u - v = x(u,v)$

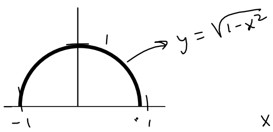
Example: Trigonometric Substitution to simplify the integration process.

what's tough is getting

$u = u(x,y)$

$v = v(x,y)$

Find the area inside half a circle of radius $r = 1$.



$x = r\cos\theta = x(r,\theta)$

$y = r\sin\theta = y(r,\theta)$

$r = r(x,y)$

$\theta = \theta(x,y)$

$\int_0^{\pi}\int_0^{1} 1\, r\,dr\,d\theta$ We need to go back the other direction.

Double integral for area in rectangular coords:

Regular $\int_{-1}^{1}\sqrt{1-x^2}\,dx =$

$$\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} 1\,dy\,dx$$

$x = r\cos\theta = 1 = \cos\theta \quad (r=1)$

$\theta = \text{Arccos}(1) = 0$

$\theta = \text{Arccos}(-1) = \pi$

is how we get it

Trig Substitution

$x = \cos\theta \implies dx = -\sin\theta\,d\theta$

$\sqrt{1-\cos^2\theta} = \sqrt{\sin^2\theta} = |\sin\theta| = \sin\theta$, provided $0 \le \theta \le \pi$

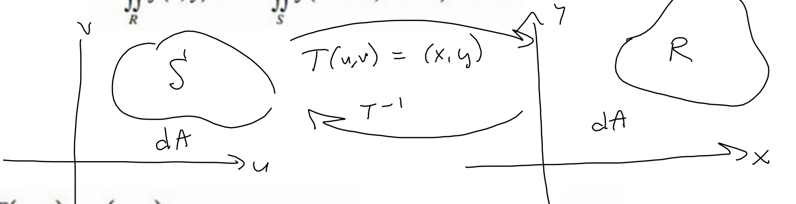
$$\int_{-1}^{-1}\sqrt{1-x^2}\,dx = \int_0^{\pi}\sin\theta\cdot(-\sin\theta\,d\theta)$$

We've already extended this to double integrals and polar coordinates:

$x = r\cos\theta$

$y = r\sin\theta$

$$\iint_R f(x,y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$



$T(u,v) = (x,y)$

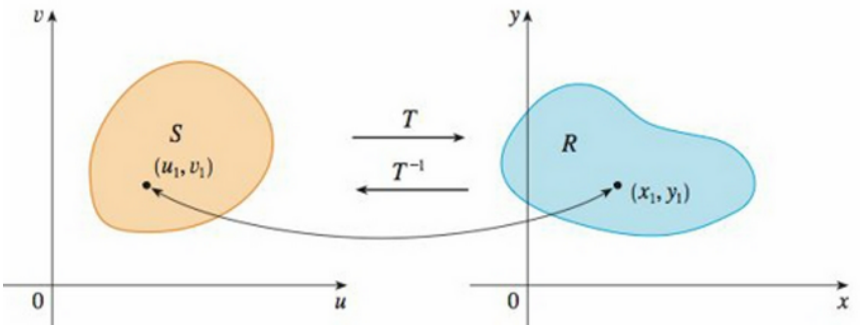
$x = g(u,v) = g(r,\theta) = r\cos\theta$ $\qquad$ $y = h(u,v) = h(r,\theta) = r\sin\theta$

$T(r,\theta) = (r\cos\theta, r\sin\theta)$

We usually assume that T is a C^1 **transformation**, which means that g and h have continuous first-order partial derivatives.

Transformations (Mappings) from one domain *onto* another.

If $T(u_1, v_1) = (x_1, y_1)$, then the point (x_1, y_1) is called the **image** of the point (u_1, v_1)



$u = G(x, y)$ $\qquad\qquad$ $v = H(x, y)$

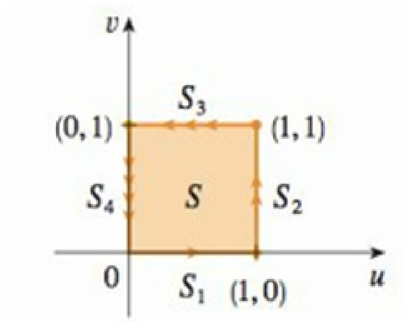
EXAMPLE 1 A transformation is defined by the equations

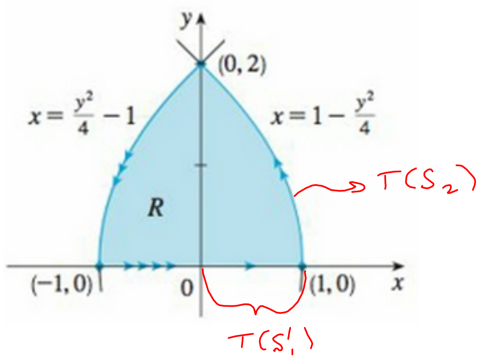
$T(x,y) = (u^2 - v^2, 2uv)$ $\quad x = u^2 - v^2 \qquad y = 2uv$

Find the image of the square $S = \{(u, v) \mid 0 \leqslant u \leqslant 1,\ 0 \leqslant v \leqslant 1\}$

SOLUTION The transformation maps the boundary of S into the boundary of the image.

is reported as a fact, with no real proof, even though it is true.





$T(S_1')$: $v = 0$, $0 \le u \le 1$

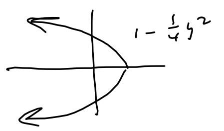
$x = u^2 - v^2 = u^2 \qquad y = 2uv = 0$

$\Rightarrow 0 \le x \le 1$

$T(S_2)$: $u = 1$, $0 \le v \le 1$

$x = 1^2 - v^2$, $y = 2v \Rightarrow v = \frac{1}{2}y$

$x = 1 - v^2 = 1 - (\frac{1}{2}y)^2 = 1 - \frac{1}{4}y^2$

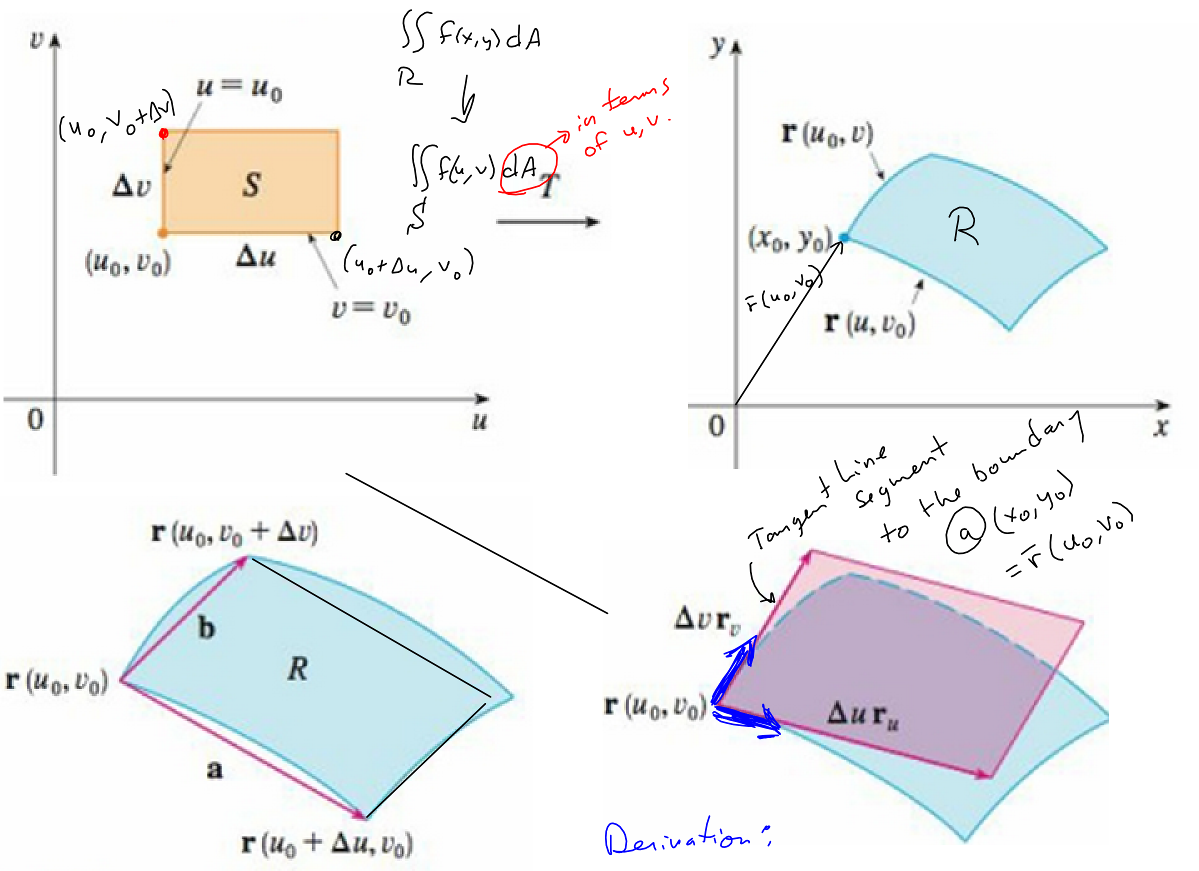


$0 \le v \le 1$

& $y = 2v \Rightarrow$

$0 \le 2v \le 2$

$0 \le y \le 2$



Area of parallelogram is $\|\bar{a} \times \bar{b}\|$

Derivation:

$$\Delta \bar{r} = \bar{b} = \bar{r}(u_0, v_0+\Delta v) - \bar{r}(u_0, v_0)$$

$$\Rightarrow \frac{\Delta \bar{r}}{\Delta v} = \frac{\bar{r}(u_0, v_0+\Delta v) - \bar{r}(u_0,v_0)}{\Delta v} \approx \frac{\partial \bar{r}}{\partial v} = \bar{r}_v$$

$$\Rightarrow \bar{b} = \bar{r}(u_0, v_0+\Delta v) - \bar{r}(u_0, v_0)$$

$$= \frac{\bar{r}(u_0, v_0+\Delta v) - \bar{r}(u_0, v_0)}{\Delta v} \cdot \Delta v$$

$$\approx \frac{\partial \bar{r}}{\partial v} \cdot \Delta v = \bar{r}_v \Delta v$$

$$\bar{a} = \frac{\partial \bar{r}}{\partial u} \cdot \Delta u = \bar{r}_u \Delta u$$

$$\Rightarrow \|\bar{a} \times \bar{b}\| \approx \|(\bar{r}_u \Delta u) \times (\bar{r}_v \Delta v)\| = \|\bar{r}_u \times \bar{r}_v\| \Delta u \Delta v = dA$$

The increment of area for the conversion!

$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Area of the region R using mapping T & domain S!

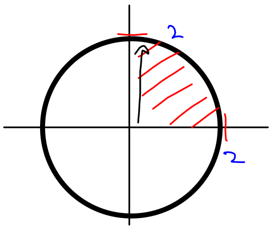
7 DEFINITION The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Double Integral in Polar Coordinates. Where did the $r\,dr\,d\theta$ come from?

We know, but here we do it in terms of the new machinery:

$$\int_0^1\int_0^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx$$



$x = r\cos\theta$, $y = r\sin\theta$

$T(r\cos\theta, r\sin\theta) = (x, y)$

$\bar{r}(r,\theta) = \langle r\cos\theta, r\sin\theta\rangle$

$\bar{r}_r = \langle \cos\theta, \sin\theta\rangle$

$\bar{r}_\theta = \langle -r\sin\theta, r\cos\theta$

$\|\bar{r}_r \times \bar{r}_\theta\|$

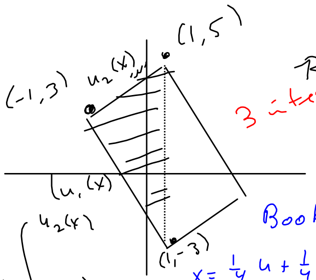
$\begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = |r\cos^2\theta + r\sin^2\theta|$

$= |r| = r$ if $0 \le r$

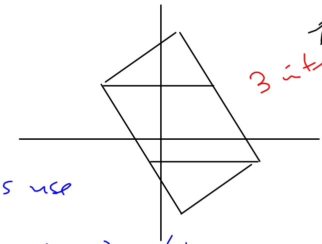
It's the r in the $r\,dr\,d\theta$!

#12

$\iint_R (4x+8y)\,dA$, where R is the parallelogram with ~~corners~~ corners $(-1,3)$, $(1,-3)$, $(3,-1)$, $(1,5)$



R
3 integrals?



R
3 integrals?

$\int_{-1}^{1}\int_{u_1(x)}^{u_2(x)}$

Book says use

$x = \frac{1}{4}u + \frac{1}{4}v$

$y = \frac{1}{4}v - \frac{3}{4}u$

$T(u,v) = \left(\frac{1}{4}u + \frac{1}{4}v,\ \frac{1}{4}v - \frac{3}{4}u\right) = (x,y)$

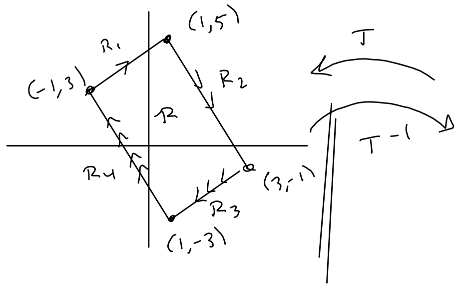
$\bar{r}(u,v) = \left\langle \frac{1}{4}u + \frac{1}{4}v,\ \frac{1}{4}v - \frac{3}{4}u \right\rangle$

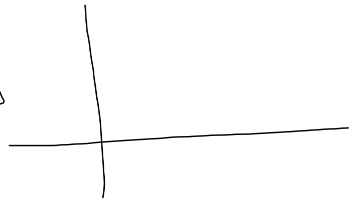
$\bar{r}_u = \left\langle \frac{1}{4}, -\frac{3}{4} \right\rangle,\quad \bar{r}_v = \left\langle \frac{1}{4}, \frac{1}{4} \right\rangle$

$\begin{vmatrix} \frac{1}{4} & -\frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} \end{vmatrix} = \frac{1}{16} + \frac{3}{16} = \frac{4}{16} = \boxed{\frac{1}{4} = \frac{\partial(x,y)}{\partial(u,v)}}$

$4x + 8y = 4\left(\frac{1}{4}u + \frac{1}{4}v\right) + 8\left(\frac{1}{4}v - \frac{3}{4}u\right)$

$= u + v + 2v - 6u = -5u + 3v$





To get T^{-1}, solve for u and v:

$x = \frac{1}{4}u + \frac{1}{4}v$

$y = \frac{1}{4}v - \frac{3}{4}u$

$\frac{1}{4}u + \frac{1}{4}v = x$

$-\frac{3}{4}u + \frac{1}{4}v = y$

$\longrightarrow \left[\begin{array}{cc|c} \frac{1}{4} & \frac{1}{4} & x \\ -\frac{3}{4} & \frac{1}{4} & y \end{array}\right]$

$T($

Reduced Row-Echelon Form: $\sim \left[\begin{array}{cc|c} 1 & 0 & x-y \\ 0 & 1 & 3x+y \end{array}\right]$

$\Rightarrow$ $u = x - y$, $v = 3x + y$

T^{-1}!

$(-1,3)$, $(1,-3)$, $(3,-1)$, $(1,5)$

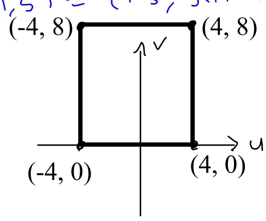
$T^{-1}(x,y) = (u,v) = (x-y,\ 3x+y)$

$T^{-1}(-1,3) = (-1-3,\ 3(-1)+3) = (-4,0)$

$T^{-1}(1,-3) = (1+3,\ 3(1)-3) = (4,0)$

$T^{-1}(3,-1) = \boxed{(3-1,\ 3(3)+1) = (2,10)}$ No. $(3 - (-1), 3(3)+(-1)) = (4, 8)$

$T^{-1}(1,5) = (1-5,\ 3(1)+5) = (-4,8)$

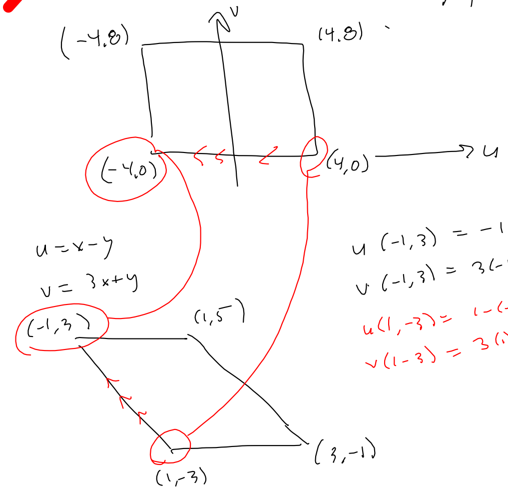


$x = \frac{1}{4}u + \frac{1}{4}v$
$y = \frac{1}{4}v - \frac{3}{4}$

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{3}{4} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} u+v \\ u-3v \end{bmatrix} = \begin{bmatrix} \frac{1}{4}u + \frac{1}{4}v \\ \frac{1}{4}u - \frac{3}{4}v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

This describes the mapping T from R in xy-plane to S in uv-plane.



$u = x - y$
$v = 3x + y$

$u(-1,3) = -1-3 = -4$
$v(-1,3) = 3(-1)+3 = 0$
$u(1,-3) = 1-(-3) = 4$
$v(1-3) = 3(1)-3 = 0$

$x = \frac{1}{4}u + \frac{1}{4}v$
$y = \frac{1}{4}u - \frac{3}{4}v$

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{3}{4} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$T(u,v) = \langle x, y \rangle$

T^{-1} is nice, but not always obtainable:

$$^{-1} = \frac{1}{4}\begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$ & it tells us where (x,y) are sent to in the uv-plane.

$$T^{-1}(T(u,v))$$

$$= T^{-1}\left(\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{3}{4} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}\right) = T^{-1}\left(\begin{bmatrix} \frac{1}{4}u + \frac{1}{4}v \\ \frac{1}{4}u - \frac{3}{4}v \end{bmatrix}\right)$$

$$= \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{4}u + \frac{1}{4}v \\ \frac{1}{4}u - \frac{3}{4}v \end{bmatrix} = \begin{bmatrix} \frac{3}{4}u + \frac{3}{4}v + \frac{1}{4}u - \frac{3}{4}v \\ \frac{1}{4}u + \frac{1}{4}v - \frac{1}{4}u + \frac{3}{4}v \end{bmatrix}$$

$$= \begin{bmatrix} u \\ v \end{bmatrix}$$

It's hard to find the inverse function, generally. Only in this case where T was linear did we have a technique that I destroyed. Generally, we can't find the inverst of the mapping T at ALL.

But the Jacobian means we don't HAVE to.

Nov 12-11:34 AM