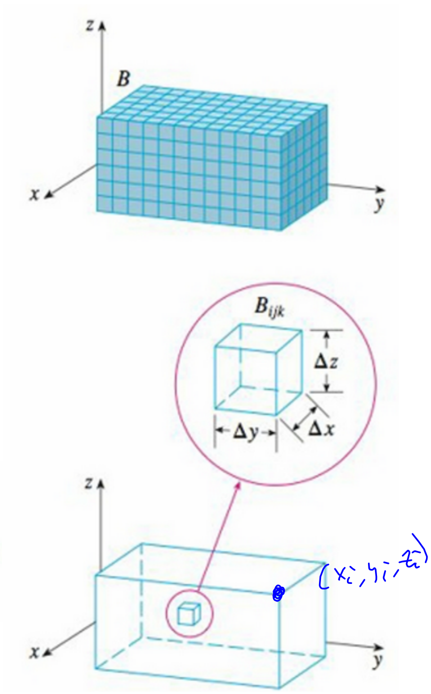
15.6 TRIPLE INTEGRALS



1-D: Break up an interval into subintervals.

2-D: Break up a rectangle into subrectangles.

3-D: Break up a box into sub-boxes.

$$B = \{(x, y, z) \mid a \leq x \leq b,\ c \leq y \leq d,\ r \leq z \leq s\}$$

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

if this limit exists. $\Delta V = \Delta x\, \Delta y\, \Delta z.$

We can simplify the writing of this as follows:

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i, y_j, z_k)\, \Delta V$$

if we choose $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) = (x_i, y_j, z_k)$

FUBINI'S THEOREM FOR TRIPLE INTEGRALS If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz = \int_a^b \int_r^s \int_c^d f(x, y, z)\, dy\, dz\, dx =$$

There are P(3,3) ways to permute the order of integration: That means 3*2*1 = 6 ways.

The triple integral over a general bounded region E

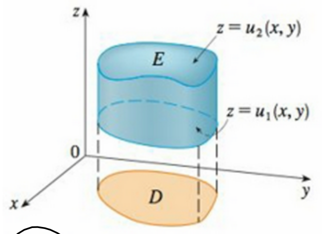


FIGURE 2
A type 1 solid region

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

Now for Type I and Type II projections beneath the Type 1 solid:

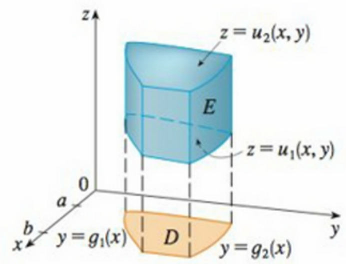
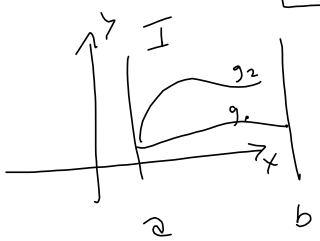


FIGURE 3
A type 1 solid region where the projection D is a type I plane region

$$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dy\, dx$$

TYPE I domain



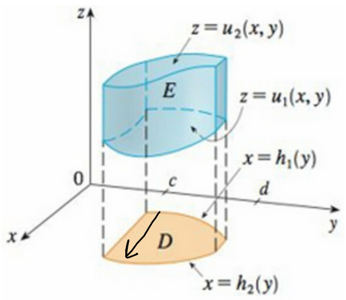


FIGURE 4
A type 1 solid region with a type II projection

$$\iiint_E f(x, y, z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dx\, dy$$

Combinations : Choose!

Permutations : Choose and Arrange.

$$P(3,3) = \frac{3!}{(3-3)!} = \frac{3!}{0!} = \frac{3 \cdot 2 \cdot 1}{1} = 6$$

xyz, xzy, yxz, yzx, zxy, zyx

$$P(n,k) = \frac{n!}{(n-k)!} \qquad C(n,k) = \frac{n!}{k!(n-k)!}$$

A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D,\ u_1(y, z) \leq x \leq u_2(y, z)\}$$

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx \right] dA$$

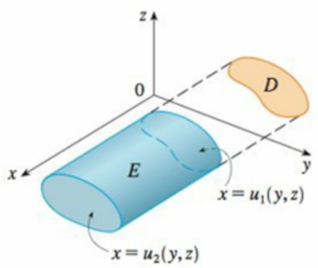


FIGURE 7 A type 2 region

A **type 3** region is of the form

$$E = \{(x, y, z) \mid (x, z) \in D,\ u_1(x, z) \leq y \leq u_2(x, z)\}$$

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy \right] dA$$

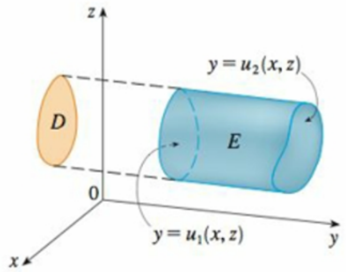
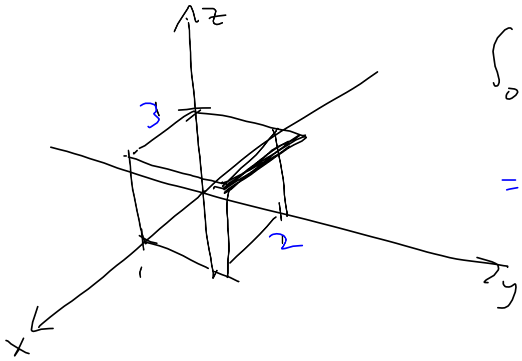


FIGURE 8 A type 3 region

$$\iiint_{\mathcal{D}} xyz \, dV$$

$$\mathcal{D} = \{(x,y,z) \mid 0 \le x \le 1, \; 0 \le y \le 2 \quad 0 \le z \le 3\}$$



$$\int_0^1 \int_0^2 \int_0^3 xyz \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^2 \left[\tfrac{1}{2} x y z^2\right]_0^3 dy \, dx$$

$$= \int_0^1 \int_0^2 \tfrac{9}{2} xy \, dy \, dx$$

$$= \frac{9}{2} \int_0^1 \left[\frac{xy^2}{2}\right]_0^2 dx$$

$$= \frac{9}{4} \int_0^1 4x \, dx$$

$$= \frac{9}{4} \left[2x^2\right]_0^1 = \boxed{\frac{9}{2}}$$

$$= \int_0^3 \int_0^2 \int_0^1 xyz \, dx \, dy \, dz$$

$$= \frac{1}{2} \int_0^3 \int_0^2 \left[x^2\right]_0^1 yz \, dy \, dz = \frac{1}{2} \int_0^3 \int_0^2 yz \, dy \, dz$$

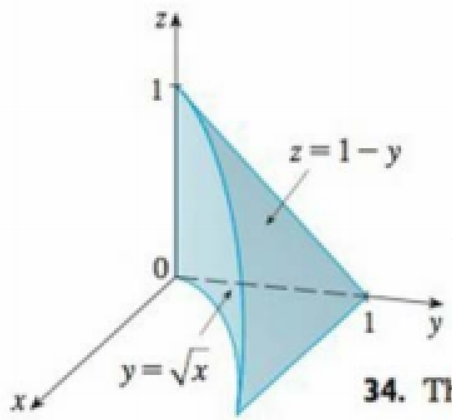
$$= \frac{1}{4} \int_0^3 \left[y^2 z\right]_0^2 dz = \int_0^3 z \, dz = \left.\frac{z^2}{2}\right]_0^3 = \boxed{\frac{9}{2}}$$

It don't matter.

33. The figure shows the region of integration for the

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx$$

Rewrite this integral as an equivalent iterated integ
five other orders.



34. The

TI over TII

I_y dz dx dy

$$\int_0^1 \int_0^{y^2} \int_0^{1-y}$$

4

dx dy dz

$$\int_0^1 \int_0^{1-z} \int_0^{y^2} dx\, dy\, dz$$

dy dz dx

$$\int_0^1 \int_0^{1-y} \int_{\sqrt{x}}^1$$