Section 15.3 Double/Iterated Integrals over General Regions

S 15.3 #s 1, 8, 15, 19, 20, 31, 48, 53 from Handout.

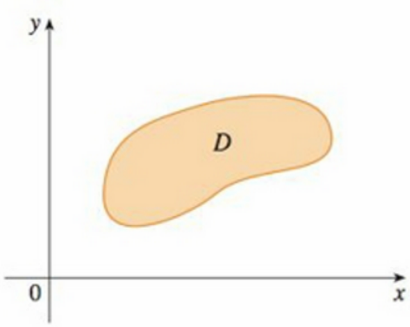


FIGURE 1

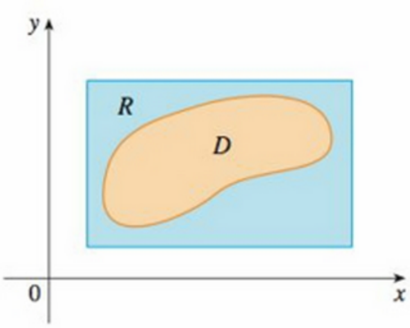
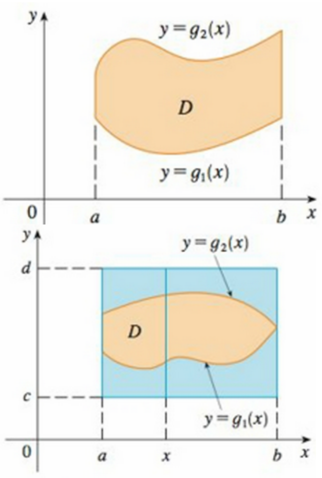


FIGURE 2

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

Then everything in 15.2 applies

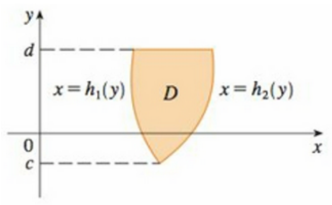
to $\iint_R F(x,y)\,dA = \iint_D f(x,y)\,dA$



TYPE I

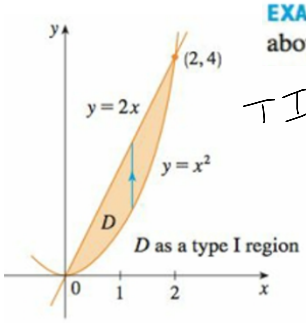
$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

TYPE II



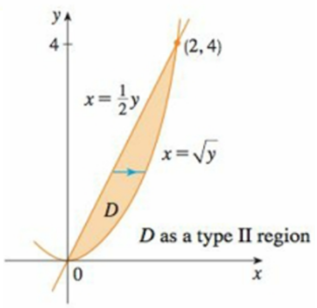
$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

EXAMPLE 2 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.



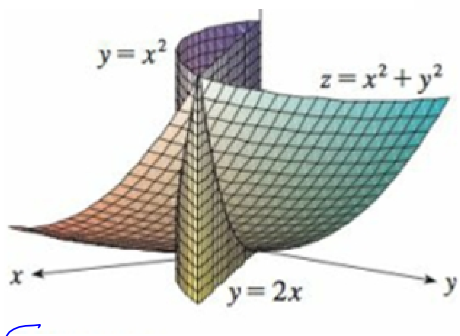
T I

$$\int_0^2 \int_{y=x^2}^{y=2x} (x^2+y^2)\,dy\,dx$$



$$\int_0^2 \int_{x=\frac{1}{2}y}^{x=\sqrt{y}} (x^2+y^2)\,dx\,dy$$

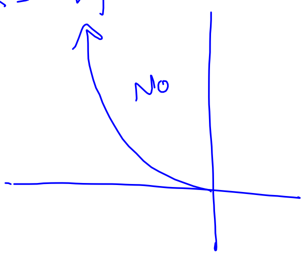
$$\int_0^4 \int_{\frac{1}{2}\cdot y}^{\sqrt{y}} (x^2+y^2)\,dx\,dy = \frac{216}{35} \quad \& \quad \int_0^2 \int_{x^2}^{2\cdot x} (x^2+y^2)\,dy\,dx = \frac{216}{35}$$



$y = 2x \Rightarrow x = \frac{1}{2}y$

$y = x^2 \Rightarrow x^2 = y \Rightarrow x = \pm\sqrt{y} \quad = +\sqrt{y}$ from the picture

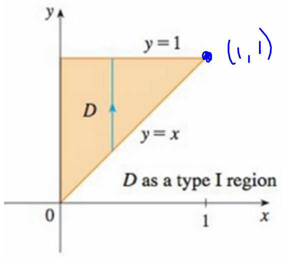
$x = -\sqrt{y}$:

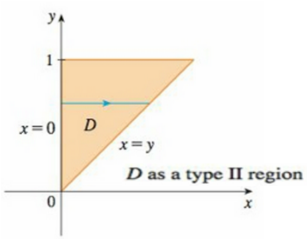


$x = +\sqrt{y}$:

EXAMPLE 5 Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$. is written as Type I over the triangle.

This one has an integrand that is not an elementary function, which is to say there's not a closed-form antiderivative for the integrand.





As a type II:

$$\int_0^1 \int_0^y \sin(y^2)\,dx\,dy$$

$$= (FYI) = \int_0^1 \sin(y^2) \int_0^y dx\,dy$$

$$= \int_0^1 \sin(y^2) \left[x \right]_0^y dy$$

$$= \int_0^1 \sin(y^2) \left[y \right] dy = \int_0^1 y \sin(y^2)\,dy$$

$$= \frac{1}{2} \int_{y=0}^{y=1} 2y \sin(y^2)\,dy = \frac{1}{2} \int_{y=0}^{y=1} \left(\sin(y^2)\right)(2y\,dy)$$

$u = y^2$

$du = 2y\,dy$

$$= \frac{1}{2} \int_{y=0}^{y=1} \sin(u)\,du$$

optional: $y = 0 \Rightarrow u = y^2 = 0$ & $y = 1 \Rightarrow u = y^2 = 1^2 = 1$

$$= \frac{1}{2} \left[-\cos(u) \right]_{y=0}^{y=1}$$

optional $= \frac{1}{2} \left[-\cos(u) \right]_{u=0}^{u=1}$

$$= \frac{1}{2} \left[-\cos(y^2) \right]_{y=0}^{y=1}$$

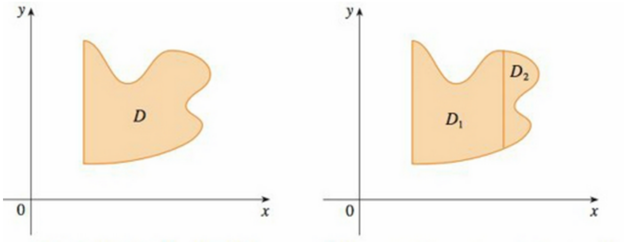
$$= \frac{1}{2} \left[-\cos(1) - (-\cos(0)) \right]$$

$$= -\frac{1}{2}\cos(1) + \frac{1}{2}$$

$$= \boxed{\frac{1}{2} - \frac{1}{2}\cos(1)}$$

$$\int f(g(x))\, g'(x)\,dx = \int f(u)\,du$$

We can partition the domain D in order to make things work.



(a) D is neither type I nor type II. (b) $D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

$$\iint_D 1\, dA = A(D)$$

If $m \leq f(x, y) \leq M$ for all (x, y) in D, then $mA(D) \leq \iint_D f(x, y)\, dA \leq MA(D)$