Section 14.4 Tangent Planes and Linear Approximations

We know from Equation 12.5.7 that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$\langle A, B, C \rangle = \vec{n}$$

$$\boxed{1} \quad z - z_0 = a(x - x_0) + b(y - y_0)$$

$$z = a(x - x_0) + b(y - y_0) + z_0$$

$$y - y_0 = m(x - x_0)$$

$$y = m(x - x_0) + y_0$$

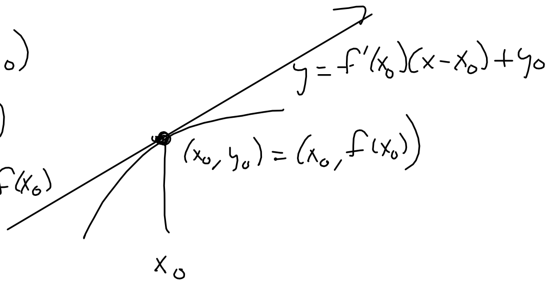
start @ (x_0, y_0) & use m to tell you how much y changes as you move away from x_0

Tangent Line

$$y = f(x_0) + f'(x_0)(x - x_0)$$

$$= f'(x_0)(x - x_0) + f(x_0)$$

Linear Approx: $L(x) = f'(x_0)(x - x_0) + f(x_0)$



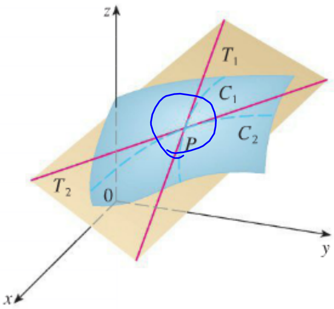
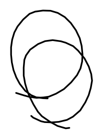


FIGURE 1
The tangent plane contains the tangent lines T_1 and T_2.

Oblate spheroid

EARTH



$\boxed{2}$ Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + z_0$$

@ (x_0, y_0), $z = z_0$

TEC Visual 14.4 shows an animation of Figures 2 and 3.

ing the domain of the function $f(x, y) = 2x^2 + y^2$. Notice that the more we zoom in, the flatter the graph appears and the more it resembles its tangent plane.

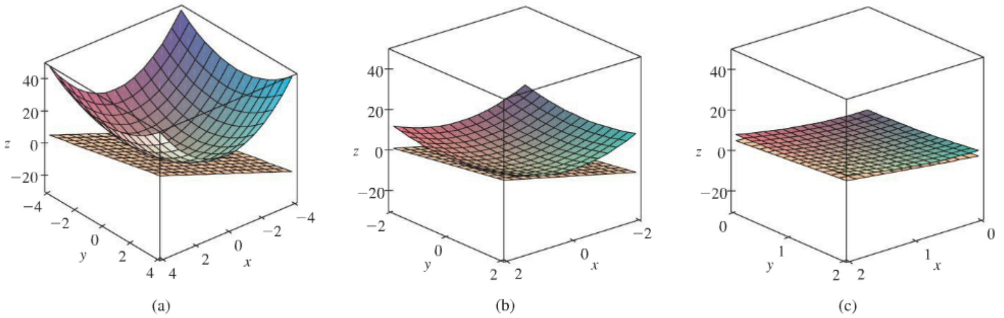
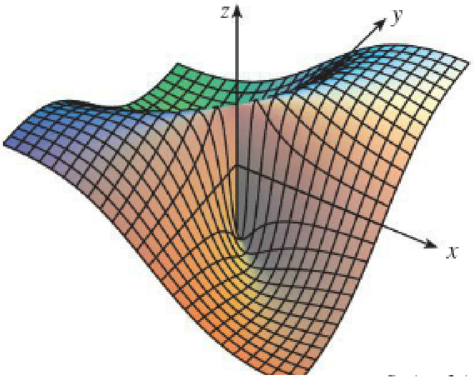


FIGURE 2 The elliptic paraboloid $z = 2x^2 + y^2$ appears to coincide with its tangent plane as we zoom in toward (1, 1, 3).

Smooth curves and smooth surfaces are LOCALLY LINEAR.



This one has a cusp at the origin, its derivatives of all orders exist, but they aren't continuous at the origin.

So a function of two variables can behave badly even though both of its partial derivatives exist.

FIGURE 4

$f(x, y) = \dfrac{xy}{x^2 + y}$ if $(x, y) \neq (0, 0)$,

$f(0, 0) = 0$

Increment of y:

$\boxed{5}$ $$\Delta y = f'(a)\,\Delta x + \varepsilon\,\Delta x \qquad \text{where } \varepsilon \to 0 \text{ as } \Delta x \to 0$$

$dy = f'(a)dx \approx \Delta y$

Increment of z:

$\boxed{6}$ $$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

$\boxed{7}$ **Definition** If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

If you want to play with these ideas (and formalisms), the #46 is the bomb.

If you *don't*, then the following is a very practical way to check for differentiability is given by:

$\boxed{8}$ **Theorem** If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

& continuity is 99% DOMAIN
(Maybe check some boundaries.)

Differentials in the Plane:

9 $$\Delta y \approx dy = f'(x)\,dx$$

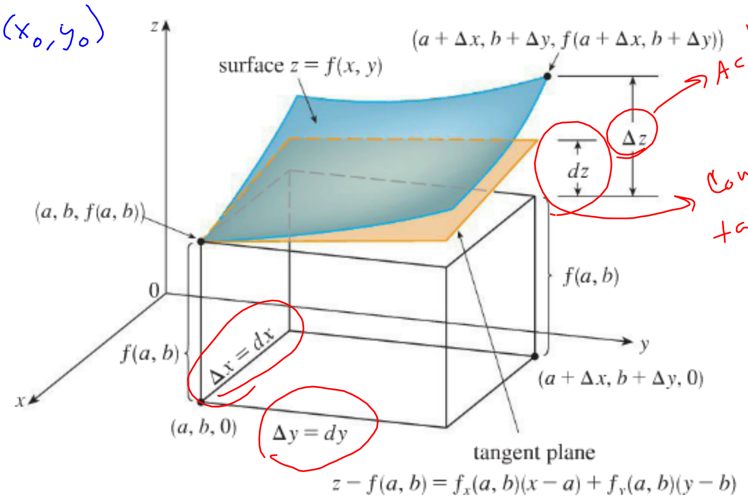
→ In practice, $dx \equiv \Delta x$

The Differential of a surface in 3-space:

10 $$\boxed{dz = f_x(x, y)\,dx + f_y(x, y)\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy}$$

Also called the "total differential."

$\Delta z = f(x, y) - f(x_0, y_0)$



→ Actual change in height

→ Comes from tangent plane

FIGURE 7

$g(x,y) = 6 - x - x^2 - 2y^2$

$g_x = -1-2x \Rightarrow g_x(1,2) = -1-2 = -3$

$g_y = -4y \Rightarrow g_y(1,2) = -8$

$g(1,2) = -4$

$L_{(1,2)}(x,y) = g_x(x-1) + g_y(y-2) - 4$

$= \boxed{-3(x-1) - 8(y-2) - 4 = L(x,y)}$