Functions of 3 or more variables...

$f(x_1, x_2, \ldots, x_n)$

f_{x_3}, etc. Same deal.

Higher Derivatives

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

f_{xy} means $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

$\frac{\partial^2 f}{\partial y \partial x}$ means the same thing.

(order is reversed from f_{xy} notation

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2}{\partial y \partial x}\left[f\right] = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$$

$$f(x,y) = x^2 \sin(xy) + x^4 y^5$$

Find $f_x(x,y)$, $f_y(x,y)$, $f_{xx}(x,y)$, $f_{yy}(x,y)$, $f_{xy}(x,y)$, $f_{yx}(x,y)$

$f := (x, y) \rightarrow x^2 \cdot \sin(x \cdot y) + x^4 \cdot y^5$

$$f := (x, y) \mapsto x^2 \sin(y\,x) + x^4 y^5$$

$fx := diff(f(x, y), x)$

$$fx := 2\,x \sin(y\,x) + x^2\,y \cos(y\,x) + 4\,x^3\,y^5$$

$fxx := diff(fx, x)$

$$fxx := 2 \sin(y\,x) + 4\,x\,y \cos(y\,x) - x^2\,y^2 \sin(y\,x) + 12\,x^2\,y^5$$

$fxy := diff(fx, y)$

$$fxy := 3\,x^2 \cos(y\,x) - x^3\,y \sin(y\,x) + 20\,x^3\,y^4$$

$fy := diff(f(x, y), y)$

$$fy := x^3 \cos(y\,x) + 5\,x^4\,y^4$$

$fyx := diff(fy, x)$

$$fyx := 3\,x^2 \cos(y\,x) - x^3\,y \sin(y\,x) + 20\,x^3\,y^4$$

$fxy - fyx$

$$0$$

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Smoothness is why.

Laplace's equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ **harmonic functions**;
heat conduction, fluid flow, and electric potential.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

"Not smooth" is rare.

$f(x) = (x-2)^{\frac{2}{5}}$

Clairaut's Theorem works 'most anywhere. Look for blowups. (Domain of f, f_x, etc)

cusp @ $x = 2$ ⤳ $(2,0)$

Not smooth @ $x = 2$