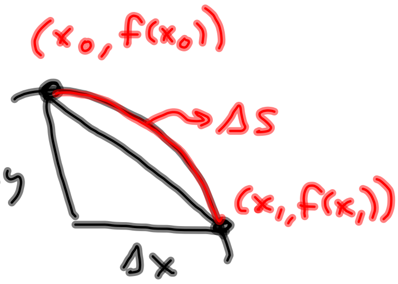
An example from §6.3 that I can actually work out w/o tech.

Arc length

$$ds = \sqrt{1+(f'(x))^2}\,dx$$



$$\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \sqrt{(\Delta x)^2 + (f(x_1) - f(x_2))^2}$$

$$= \sqrt{(\Delta x)^2 + (f'(c_k)\,\Delta x)^2}$$

by MVT $f'(c_k) = \frac{f(x_1)-f(x_0)}{x_1 - x_0} = \frac{\Delta y}{\Delta x}$

$$= \sqrt{1 + (f'(c_k))^2}\ \underbrace{\Delta x}_{\Delta x > 0}$$

$$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

$$L(x) = \int_a^x \sqrt{1+(f'(t))^2}\,dt$$

An increasing function of x.
as $x \to$ Bigger, this grows.

This is handy for when you have many arc lengths to figure for same function.

$L(x) = \int_0^x$, $L(2) = \int_0^2$, $L(3) = \int_0^3$, $L(7) = \int_0^7$

$ds = \sqrt{1+(f'(x))^2}\,dx$ an increment of arc length

$L = \int ds$

$y = f(x) = \frac{1}{3}(x^2+2)^{\frac{3}{2}}$ $\quad 0 \le x \le 3$

Find arc length.

$\int_0^3 \sqrt{1+(f'(x))^2}\,dx$

$= \int_0^3 \sqrt{1+(y')^2}\,dx$

$= \int_0^3 ds$

$f'(x) = x(x^2+2)^{\frac{1}{2}}$

$f'(x)^2 = x^2(x^2+2)$

$= x^4+2x^2$

$1+f'(x)^2 = x^4+2x^2+1$

$= (x^2+1)^2$

$$\int_0^3 \sqrt{1+(f'(x))^2}\,dx = \int_0^3 \sqrt{(x^2+1)^2}\,dx = \int_0^3 |x^2+1|\,dx$$

$$= \int_0^3 (x^2+1)\,dx = \left[\frac{x^3}{3} + x\right]_0^3 = \frac{3^3}{3}+3-\left(\frac{0^3}{3}+0\right)$$

$x^2+1 \ge 0$ $\quad = 12$

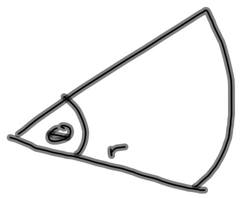
$$L(x) = \int_0^x \sqrt{1+(f'(t))^2}\,dt = \left[\frac{t^3}{3}+t\right]_0^x$$

$$= \frac{x^3}{3}+x-\left(\frac{0^3}{3}+0\right) = \frac{x^3}{3}+x$$

Arc length for $0 \le x \le 7$ is $L(7) = \frac{7^3}{3}+7$

$L(13) = \frac{13^3}{3}+13$ goes quick.

area of circular sector swept by an angle θ, if radius = r.



$\theta = \underline{2\pi} = \frac{2\pi}{1}$

$\underline{A = \pi r^2}$

$= \underline{\frac{1}{2}(2\pi)r^2}$

$\theta = \pi$

$\frac{\pi}{2} r^2$ $\frac{1}{2}$-circle

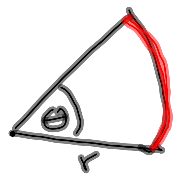
$\theta = \frac{\pi}{4} = \frac{2\pi}{2\cdot 4} = \frac{2\pi}{8}$

So $A = \underline{\frac{\pi r^2}{8}}$

See the proportion?

$\frac{1}{2} r^2 \theta$

How about arc length?



$2\pi r$ all the way around

θr when $\theta = 2\pi$

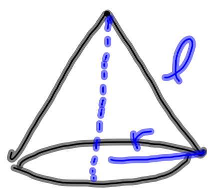
$s =$ $\theta = \pi$

$s = \pi r = r\pi = r\theta =$ arc length.

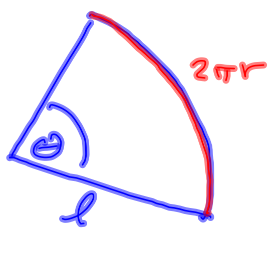
Think of this proportion:

Cone: radius r

side length ℓ



area of cone is area of this circular sector:



$=$

$\boxed{\ell\theta} = 2\pi r$

$\frac{\theta}{2\pi} = \frac{2\pi r}{2\pi \ell}$

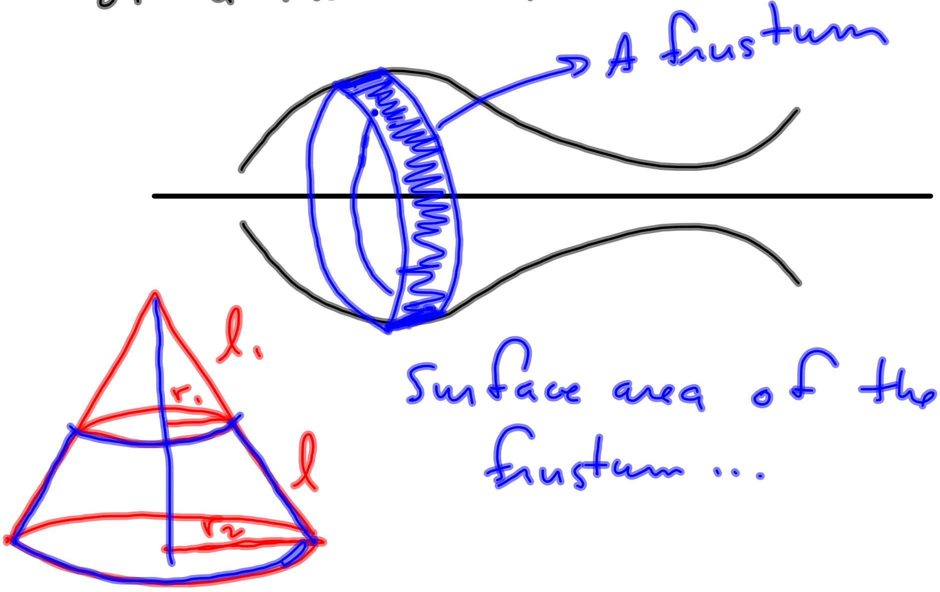
$\frac{\theta}{2\pi} = \frac{r}{\ell}$

$\theta = \frac{2\pi r}{\ell}$ $\Longrightarrow$ $\frac{1}{2}\ell^2\theta = \text{Area}$

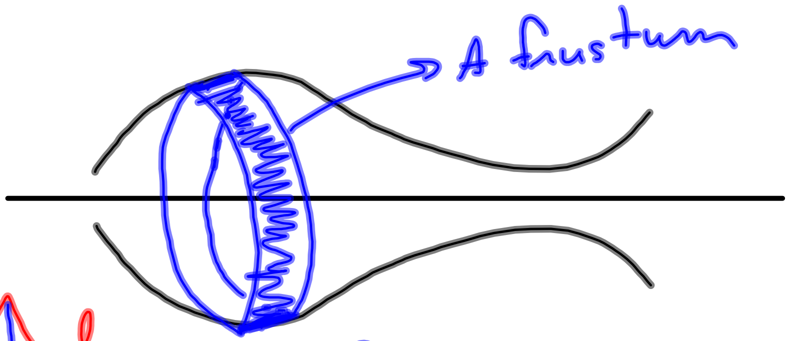
$= \frac{1}{2}\ell^2\left(\frac{2\pi r}{\ell}\right)$

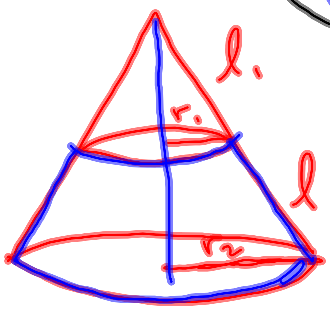
$= \pi r \ell =$ surface area of the cone!

Our Goal: to find surface area of a solid of revolution. We use surface area of a FRUSTUM.



Surface area of the frustum ...





Surface area of the frustum ...

= BIG CONE − LITTLE CONE

$$= \pi r_2(\ell+\ell_1) - \pi r_1 \ell_1$$

By similar triangles $\frac{\ell_1}{r_1} = \frac{\ell_1+\ell}{r_2}$

$$r_2\ell_1 = r_1(\ell_1+\ell) = r_1\ell_1 + r_1\ell$$

$$r_2\ell_1 - r_1\ell_1 = r_1\ell$$

$$(r_2-r_1)\ell_1 = r_1\ell \implies$$

$$\pi(r_2-r_1)\ell_1 = \pi r_1\ell$$

$$\underbrace{\pi(r_2-r_1)\ell_1 + \pi r_2\ell}_{\text{Area of Frustum } \pi r_2(\ell_1+\ell_2) - \pi r_1\ell_1} = \pi r_1\ell + \pi r_2\ell$$

$$= \pi(r_1+r_2)\ell$$

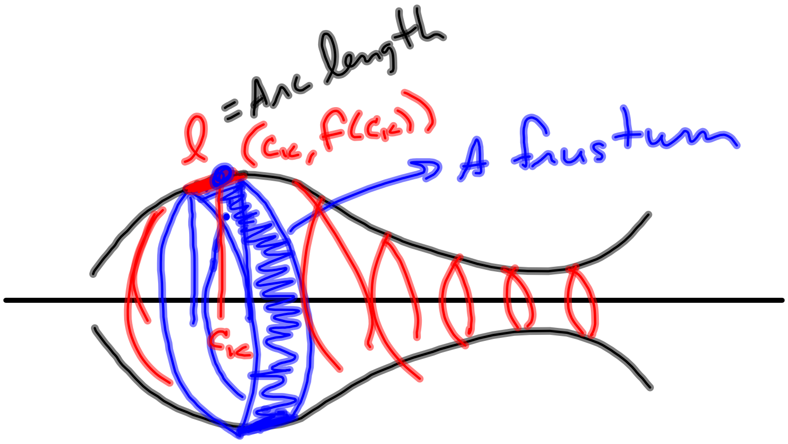
$$= 2\pi\Big(\underbrace{\frac{r_1+r_2}{2}}_{\text{Average r-value.}}\Big)\ell$$

$= 2\pi r\ell$, where

r = Average radius

This r will be $\frac{f(x_1)+f(x_2)}{2}$ in the sequel

& IVT says $f(c_k) = \frac{f(x_1)+f(x_2)}{2}$



$2\pi r l$

Area = $2\pi r l$ = $2\pi f(c_k)\sqrt{1+f'(c_k)^2}\,dx$

Leads us to $2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx$