201-G11 Worksheet 5.1 – 5.4
Fall, 2012 Name_______________________

Evaluate the following definite integrals in two ways:

I. As the limit of a Riemann sum, using right endpoints. (S 5.2)
II. By the Fundamental Theorem of Calculus, Part 2 (S 5.4)

Any time you can use previous work, you *should*, and just refer the reader back to it.

1. $\int_1^3 3dx$ $\quad \Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$

$x_k = a + k\Delta x = 1 + \frac{2k}{n} = \frac{2k+n}{n}$

$$\sum_{k=1}^{n} f(x_k)\Delta x_k = \sum_{k=1}^{n} 3 \cdot \frac{2}{n} = \frac{6}{n}\sum_{k=1}^{n} 1 = \frac{6}{n}\cdot n = 6$$

$$\int_1^3 3dx = \left[3x\right]_1^3 = 3(3) - 3(1) = 9 - 3 = 6$$

2. $\int_1^3 4xdx$ $= f(x_k) = 4\left(\frac{2k+n}{n}\right) \Longrightarrow$

$$\sum_{k=1}^{n} f(x_k)\Delta x_k = \sum_{k=1}^{n} 4\left(\frac{2k+n}{n}\right)\cdot\frac{2}{n} = \frac{8}{n^2}\sum_{k=1}^{n}(2k+n)$$

$$= \frac{8}{n^2}\left[\sum_{k=1}^{n} 2k + \sum_{k=1}^{n} n\right] = \frac{16}{n^2}\sum_{k=1}^{n} k + \frac{8}{n}\sum_{k=1}^{n} 1 \xrightarrow{n \to \infty}$$

$$= \frac{16}{n^2}\left(\frac{n^2 + \text{small}}{2}\right) + \frac{8}{n}\cdot n = 8 + 8 = 16$$

$$\int_1^3 4x\,dx = \left[\frac{4x^2}{2}\right]_1^3 = \left[2x^2\right]_1^3 = 2(9) - 2(1) = 18 - 2 = 16$$

3. $\int_1^3 x^2 dx$ $f(x_k) = x_k^2 = \left(1+\frac{2k}{n}\right)^2 = 1^2 + 2(1)\left(\frac{2k}{n}\right) + \left(\frac{2k}{n}\right)^2$

$= 1 + \frac{4k}{n} + \frac{4k^2}{n^2}$

$$\sum_{k=1}^{n} f(x_k)\Delta x_k = \sum_{k=1}^{n}\left(1+\frac{4k}{n}+\frac{4k^2}{n^2}\right)\cdot\frac{2}{n}$$

$$= \frac{2}{n}\left[\sum 1 + \sum \frac{4k}{n} + \sum \frac{4k^2}{n^2}\right]$$

$$= \frac{2}{n}\left[\sum_{k=1}^{n} 1 + \frac{4}{n}\sum_{k=1}^{n} k + \frac{4}{n^2}\sum_{k=1}^{n} k^2\right]$$

$$= \frac{2}{n}\left[n + \frac{4}{n}\cdot\frac{n^2+\cdots}{2} + \frac{4}{n^2}\cdot\frac{2n^3+\cdots}{6}\right]$$

$$= \frac{2}{n}\cdot n + \frac{2}{n}\cdot\frac{4}{n}\cdot\frac{(n^2+\cdots)}{2} + \frac{2}{n}\cdot\frac{4}{n^2}\cdot\frac{2n^3+\cdots}{6}$$

$$= 2 + \frac{4n^2+\cdots}{n^2} + \frac{8n^3+\cdots}{3n^3} \xrightarrow{n\to\infty} 2+4+\frac{8}{3} = \frac{26}{3}$$

$$\int_1^3 x^2 dx = \left[\frac{x^3}{3}\right]_1^3 = \frac{27}{3} - \frac{1}{3} = \frac{26}{3} \checkmark$$

4. $\int_1^3 (x^2 - 4x + 3)dx$ $= \int_1^3 x^2 dx - \int_1^3 4x\, dx + \int_1^3 3\, dx$

$$= \frac{26}{3} - 16 + 6 = \frac{26}{3} - 10 = \frac{26-30}{3} = \boxed{-\frac{4}{3}}$$

By #s 1–3.

5. What is the average value of $f(x) = x^2 - 4x + 3$ on the interval $[1,3]$?

$$\frac{1}{3-1}\int_1^3 f(x)dx = \frac{1}{2}\cdot\left(-\frac{4}{3}\right) = -\frac{2}{3}$$

6. What value c satisfies the conclusion of the Mean Value Theorem for Integrals for $f(x) = x^2 - 4x + 3$ on the interval $[1,3]$?

$x^2 - 4x + 3 = -\frac{2}{3}$

$x^2 - 4x = -3 - \frac{2}{3} = -\frac{11}{3}$

$x^2 - 4x + 2^2 = -\frac{11}{3} + 4 = \frac{1}{3}$

$(x-2)^2 = \frac{1}{3}$

$x - 2 = \pm\sqrt{\frac{1}{3}} = \pm\frac{1}{\sqrt{3}}$ → Good enough

$\boxed{x = 2 \pm \frac{1}{\sqrt{3}}}$ →

$x = 2 + \frac{1}{\sqrt{3}} = 2 + \frac{\sqrt{3}}{3} = \boxed{\frac{6+\sqrt{3}}{3} = x}$

$x^2 - 4x + 3 = -\frac{2}{3}$

$x^2 - 4x + \frac{9}{3} + \frac{2}{3} = 0$

$x^2 - 4x + \frac{11}{3} = 0$

$b^2 - 4ac = (-4)^2 - 4(1)\left(\frac{11}{3}\right)$

$= 16 - \frac{44}{3} = \frac{48-44}{3} = \frac{4}{3}$

$x = \frac{4 \pm \sqrt{\frac{4}{3}}}{2(1)} = \frac{4 \pm \frac{2}{\sqrt{3}}}{2} = \frac{4 \pm \frac{2\sqrt{3}}{3}}{2} = \frac{\frac{12 \pm 2\sqrt{3}}{3}}{2}$

$= \frac{1}{2}\left(\frac{12 \pm 2\sqrt{3}}{3}\right) = \frac{1}{2}\left(\frac{2(6 \pm \sqrt{3})}{3}\right) = \boxed{\frac{6 \pm \sqrt{3}}{3}}$ ✓

So $\frac{6+\sqrt{3}}{3}$ is the one.