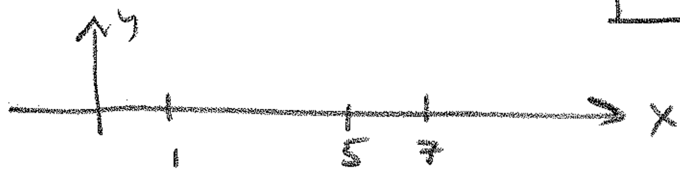
201 §2.3 #s 1, 3, 5–7, 11, 15, 17, 19, 21, 23, 25, 27, 31, 33, 35, 37, 39, 41, 43, 45, 55, 56, 57, 59

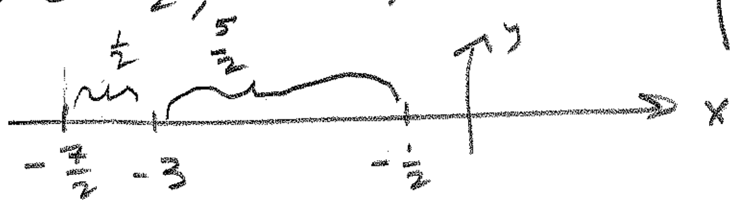
#s 1–6 Sketch interval (a,b) w/ $x_0 \in (a,b)$ on the x-axis. Find $\delta > 0 \ni \forall x$, with $0 < |x - x_0| < \delta$ $\Rightarrow a < x < b$

(1) $a = 1,\ b = 7,\ x_0 = 5$ $\boxed{\delta = 2 \text{ works}}$



JUST PICK THE SMALLER DISTANCE!

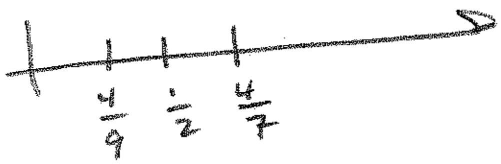
(3) $a = -\frac{7}{2},\ b = -\frac{1}{2},\ x_0 = -3$ $\boxed{\delta = \frac{1}{2}}$



(5) $a = \frac{4}{9},\ b = \frac{4}{7},\ x_0 = \frac{1}{2}$

$\frac{4}{7} - \frac{1}{2} = \frac{8-7}{14} = \frac{1}{14}$

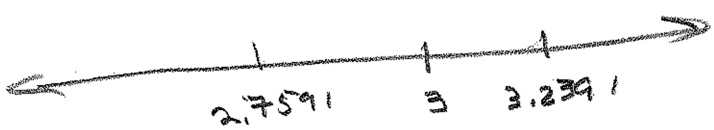
$\frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \boxed{\frac{1}{18} = \delta}$



(6) $a = 2.7591,\ b = 3.2391,\ x_0 = 3$

$3.2391 - 3 = \boxed{.2391 = \delta}$

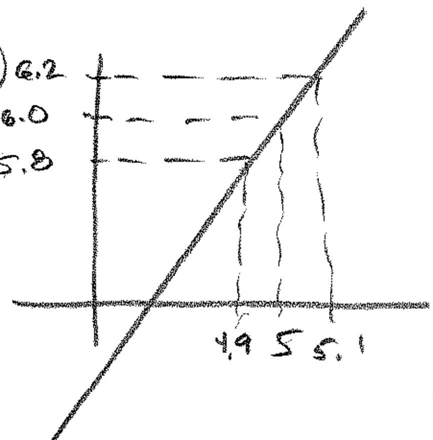
$3 - 2.7591 = .2409$



201 § 2.3 #s 7, 11, 15, 17, 19, 21, 23, 25, 27, 31, 33, 35, 37, 39, 41, 43, 45, 55, 56, 57, 59

~~#s~~ 7–14 use the graphs to find $\delta > 0$ $\ni$ $\forall$ x with $0 < |x - x_0| < \delta$, we have $|f(x) - L| < \varepsilon$

7



$f(x) = 2x - 4$

$x_0 = 5$

$L = 6$

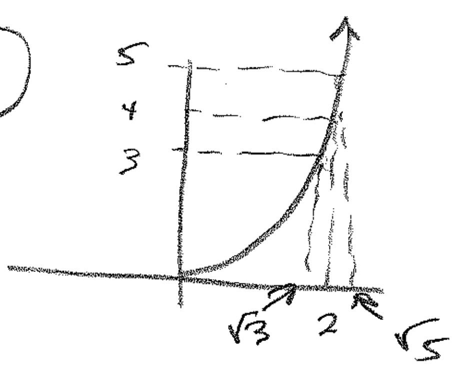
$\varepsilon = 0.2$

$\boxed{\text{Let } \delta = 0.1}$

Check: If $0 < |x-5| < \delta$, then

$|f(x) - L| = |2x - 4 - 6| = |2x - 10| = 2|x-5| < 2(.1) = .2$

11



$f(x) = x^2$

$x_0 = 2$

$\varepsilon = 1$

Scratch:

$\sqrt{5} - 2 \approx .236$

$2 - \sqrt{3} \approx .267949$

Let $\boxed{\delta = .236}$

Check: $0 < |x-2| < \delta \longrightarrow$

$|f(x) - L| = |x^2 - 4| = |x-2||x+2| < \delta|x+2|$

$< \delta|2.236 + 2| = 4.236\,\delta = .999696 < \varepsilon$

How? If I know $\delta = .236$, then x is in $(2 - .236, 2 + .236)$ and $2 + .236 = 4.236$ makes $|x+2|$ as "bad" (big) as it can get.

201 §2.3 #s 15, 17, 19, 21, 23, 25, 27, 31, 33, 35, 37, 39, 41, 43, 45, 55, 56, 57, 59

#s 15-30 Find an open interval around x_0 $\ni$ $|f(x)-L|<\varepsilon$ holds. Then find a $\delta>0$ that'll work $\forall$ x $\ni$ $0<|x-x_0|<\delta$.

15 $f(x)=x+1$, $L=5$, $x_0=4$, $\varepsilon=0.01$

want $|x+1-5|<.01$ $\Rightarrow$

$|x-4|<.01$ $\Rightarrow$ $\boxed{(a,b)=(3.99, 4.01)}$ and

we want $\boxed{\delta=0.01}$

17 $f(x)=\sqrt{x+1}$, $L=1$, $x_0=0$, $\varepsilon=0.1$

want $|\sqrt{x+1}-1|<\varepsilon=0.1$ $\Rightarrow$

$\sqrt{x+1}-1<.01$ and $\sqrt{x+1}-1>-0.01$

$\Rightarrow \sqrt{x+1}<1.01$ and $\sqrt{x+1}>.99$

$\Rightarrow x+1<1.0201$ and $x+1>.9801$

$\Rightarrow x<.0201$ and $x>-.0199$ $\longrightarrow$

$\boxed{(a,b)=(-.0199, .0201)}$. For δ, we find the minimum "radius." $\boxed{\delta=.0199}$ should work!

201 §2.3 #s 19, 21, 23, 25, 27, 31, 33, 35, 37, 39, 41, 43, 45, 55, 56, 57, 59

(19) $f(x) = \sqrt{19-x}$, $L = 3$, $x_0 = 10$, $\varepsilon = 1$

Want $|\sqrt{19-x} - 3| < 1 \longrightarrow$

$-1 < \sqrt{19-x} - 3 < 1 \longrightarrow$

$2 < \sqrt{19-x} < 4 \longrightarrow$

$4 < 19-x < 16 \longrightarrow$

$-15 < -x < -3 \longrightarrow$

$15 > x > 3 \longrightarrow$

$\boxed{(a,b) = (3,15)}$ Now with $x_0 = 10$, the smallest distance from 10 to then endpoints is 5,

so, let $\boxed{\delta = 5}$

(21) $f(x) = \frac{1}{x}$, $L = \frac{1}{4}$, $x_0 = 4$, $\varepsilon = 0.05$

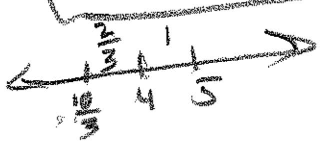
want $\left|\frac{1}{x} - \frac{1}{4}\right| < 0.05 \longrightarrow$

$-.05 < \frac{1}{x} - \frac{1}{4} < .05 \longrightarrow$

$\frac{1}{5} < \frac{1}{x} < \frac{3}{10}$

$\longrightarrow 5 > x > \frac{10}{3}$

$\longrightarrow \boxed{(a,b) = \left(\frac{10}{3}, 5\right)}$



NOW, assume $\delta < 1$ (or at least small enough to keep $x > 0$) since we're looking at $x_0 = 4$, this is safe.

Let $\boxed{\delta = \frac{2}{3}}$

201 §2.3 ~~#5~~ 23, 25, 27, 31, 33, 35, 37, 39,

41, 43, 45, 55-57, 59

(23) $f(x) = x^2$, $L = 4$, $x_0 = -2$, $\varepsilon = 0.5$

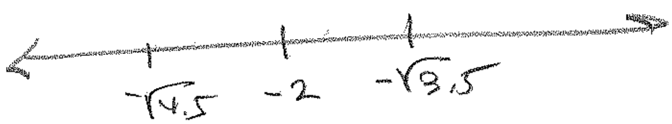
want $|x^2 - 4| < 0.5 \Rightarrow$

$-.5 < x^2 - 4 < 0.5 \Rightarrow$

$3.5 < x^2 < 4.5$

$\sqrt{3.5} < |x| < \sqrt{4.5} \Rightarrow -\sqrt{4.5} < x < -\sqrt{3.5}$

since $x_0 = -2 < 0$



$-2 - (-\sqrt{4.5}) \approx .1213\,2 < .1292 \approx -\sqrt{3.5} - (-2)$

$\boxed{\text{Let } \delta = \sqrt{4.5} - 2}$

(25) $f(x) = x^2 - 5$ is similar

(27) $f(x) = mx$, $m > 0$, $L = 2m$, $x_0 = 2$, $\varepsilon = 0.03$

Let $\boxed{\delta = \frac{\varepsilon}{m} = \frac{.03}{m}}$

~~#5~~ 31 → 36 Find $\lim_{x \to x_0} f(x) = L$ and $\delta > 0 \ni$

$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$

(31) $f(x) = 3 - 2x$, $x_0 = 3$, $\varepsilon = .02$

$\lim_{x \to 3} f(x) = \boxed{-3 = L}$, $\delta = \frac{\varepsilon}{2} = \frac{.02}{2} = \boxed{.01 = \delta}$

201 §2.3 #s 33, 35, 37, 39, 41, 43, 45, 55-57, 59

33 $f(x) = \frac{x^2-4}{x-2}$, $x_0 = 2$, $\varepsilon = 0.05$

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2 \xrightarrow{x \to 2} \boxed{4 = L}$$

$$\left| \frac{x^2-4}{x-2} - 4 \right| < .05$$

$$|x+2-4| < .05$$

$$-.05 < x-2 < .05$$

$$1.95 < x < 2.05$$

$$\boxed{\text{Let } \delta = .05}$$

35 $f(x) = \sqrt{1-5x}$, $x_0 = -3$, $\varepsilon = .5$

$.5 - 4 = -3.5 = -\frac{7}{2}$

$.5 + 4 = 4.5 = \frac{9}{2}$

$$\sqrt{1-5x} \xrightarrow{x \to -3} \sqrt{16} = \boxed{4 = L}$$

$$\left| \sqrt{1-5x} - 4 \right| < \varepsilon$$

$$-\varepsilon < \sqrt{1-5x} - 4 < \varepsilon$$

$$4-\varepsilon < \sqrt{1-5x} < \varepsilon + 4$$

$$(\varepsilon-4)^2 < 1-5x < (\varepsilon+4)^2$$

$$(\varepsilon-4)^2 - 1 < -5x < (\varepsilon+4)^2 - 1$$

$$\frac{(\varepsilon-4)^2-1}{-5} > x > \frac{(\varepsilon+4)^2-1}{-5}$$

$$\frac{1-(\varepsilon-4)^2}{5} > x > \frac{1-(\varepsilon+4)^2}{5}$$

$$\frac{1-\left(\frac{7}{2}\right)^2}{5} > x > \frac{1-\left(\frac{9}{2}\right)^2}{5}$$

$$\frac{4-49}{20} > x > \frac{4-81}{20}$$

$$-\frac{45}{20} > x > -\frac{77}{20}$$

$$-2.25 > x > -3.85$$

$.75, .85$

$$\boxed{\text{Let } \delta = .75}$$

201 §2.3 #s 37, 39, 41, 43, 45, 55–57, 59

#s 37–50 Prove the statement.
I'll use the book method.

37) $\lim_{x\to 4} (9-x) = 5$. Let $\varepsilon > 0$.

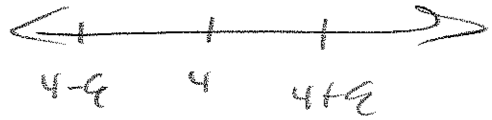
1) $|9 - x - 5| < \varepsilon$

$|4-x| < \varepsilon$

$-\varepsilon < 4-x < \varepsilon$

$-\varepsilon - 4 < -x < \varepsilon - 4$

$\varepsilon + 4 > x > 4 - \varepsilon$



2) $\boxed{\text{Let } \delta = \varepsilon}$

39) $\lim_{x\to 9} \sqrt{x-5} = 2$ Let $\varepsilon > 0$

1) $|\sqrt{x-5} - 2| < \varepsilon$

$-\varepsilon < \sqrt{x-5} - 2 < \varepsilon$

$2 - \varepsilon < \sqrt{x-5} < \varepsilon + 2$

$(\varepsilon - 2)^2 < x - 5 < (\varepsilon + 2)^2$

$(\varepsilon - 2)^2 + 5 < x < (\varepsilon + 2)^2 + 5$

2) $0 < |x - 9| < \delta$

$-\delta < x - 9 < \delta$

$9 - \delta < x < 9 + \delta$

201 §2.3 #s 39, 41, 43, 45, 55-57, 59

39 cont'd

SET $9-\delta = (\varepsilon-2)^2+5$ & $9+\delta = (\varepsilon+2)^2+5$

$-\delta = (\varepsilon-2)^2-4$

$\delta = 4-(\varepsilon-2)^2$

$\delta = 4-\varepsilon^2+4\varepsilon-4$

$\delta = 4\varepsilon-\varepsilon^2$

$\delta = (\varepsilon+2)^2-4$

$\delta = \varepsilon^2+4\varepsilon+4-4$

$\delta = \varepsilon^2+4\varepsilon$

Let $\boxed{\delta = 4\varepsilon-\varepsilon^2}$

This is all the book's asking for.
See next page and 8/28 notes for how an analyst would write a formal proof, and make estimates.

It's the stuff of higher mathematics.
It's not something we need to sweat, but it's GOOD to be exposed to this sort of thinking. It can give insight.

Extra Bonus for §2.3. How I'd write up a proof & Find a workable δ.

① Manipulate $|f(x)-L|$ to get $|x-x_0|\cdot$something. Then set $\delta = \frac{\varepsilon}{\text{something}}$.

② Write up the proof.

Scratch for #39

want $|\sqrt{x-5}-2| < \varepsilon$. Well, let's assume $\delta \le 1$, $\varepsilon < 1$. Now

$$|\sqrt{x-5}-2| = \left| \frac{(\sqrt{x-5}-2)(\sqrt{x-5}+2)}{\sqrt{x+5}+2} \right|$$

$$= \frac{|x-5-4|}{\sqrt{x+5}+2}^{*}$$. Now, $\delta \le 1 \Rightarrow 8 \le x \le 10$, so

$\sqrt{8+5}+2 \le \sqrt{x+5}+2 \le \sqrt{10+5}+2$

$\sqrt{3}+2 < \sqrt{x+5}+2 < \sqrt{5}+2$

& $\sqrt{3}+2 > 1+2 = 3$. To make $\frac{|x-a|}{\sqrt{x-5}+2}$ bigger, make its denom. smaller. ~~See 8/28 notes for the finish.~~

201 §2.3 #s 41, 43, 45, 55–57, 59

39 cntd

$$\frac{|x-9|}{\sqrt{x-5}+2} \leq \frac{|x-9|}{\sqrt{3}+2} < \frac{|x-9|}{3} < \frac{\delta}{3} \underset{\leq}{\text{want}} \varepsilon,$$

so define $\delta = 3\varepsilon$. Now write the proof.

$\boxed{\text{Pf}}$ Let $\varepsilon > 0$ be given. Define $\delta = \min\{1, 3\varepsilon\}$.

Then $0 < |x-9| < \delta \implies$

$$|\sqrt{x-5}-2| = \left|(\sqrt{x-5}-2)\left(\frac{\sqrt{x-5}+2}{\sqrt{x-5}+2}\right)\right|$$

$$= \left|\frac{x-5-4}{\sqrt{x-5}+2}\right| = \frac{|x-9|}{\sqrt{x-5}+2} < \frac{|x-9|}{3} < \frac{\delta}{3} \leq \frac{3\varepsilon}{3} = \varepsilon \blacksquare$$

41 $\lim_{x\to 1} f(x) = 1$, if $f(x) = \begin{cases} x^2 & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$

BOOK WAY

① $|x^2 - 1| < \varepsilon$

$-\varepsilon < x^2 - 1 < \varepsilon$

$1-\varepsilon < x^2 < 1+\varepsilon$

$\sqrt{1-\varepsilon} < |x| < \sqrt{1+\varepsilon}$ Assume $\delta < 1$ and $\varepsilon < 1$

Then $1-\varepsilon > 0$ & $|x| = x$, from $\delta < 1$ & $x_0 = 1$

$\sqrt{1-\varepsilon} < x < \sqrt{1+\varepsilon}$

201 § 2.3 #s 41, 43, 45, 55-57, 59

(41) cont'd

(2) $|x-1| < \delta$

$-\delta < x-1 < \delta$

$1-\delta < x < 1+\delta$

SET $1-\delta = \sqrt{1-\varepsilon}$ $\Longrightarrow$ SET $1+\delta = \sqrt{1+\varepsilon}$ $\Longrightarrow$

$-\delta = \sqrt{1-\varepsilon} - 1$ $\qquad$ $\delta = \sqrt{1+\varepsilon} - 1$

$\delta = 1 - \sqrt{1-\varepsilon}$

Without knowing ε, it's hard to say which of these is smaller, so define

$\boxed{\delta = \min\{1-\sqrt{1-\varepsilon}, \sqrt{1+\varepsilon}-1\}}$ The $f(1)=2$ had NOTHING TO DO WITH IT!

(43) $\lim_{x \to 1} \frac{1}{x} = 1$

(1) $\left|\frac{1}{x} - 1\right| < \varepsilon$

$-\varepsilon < \frac{1}{x} - 1 < \varepsilon$

$1-\varepsilon < \frac{1}{x} < 1+\varepsilon$ Assume $\delta < 1$. Then

Assume $\varepsilon < 1$

$\frac{1}{1-\varepsilon} > x > \frac{1}{1+\varepsilon}$

$\frac{1}{1+\varepsilon} < x < \frac{1}{1-\varepsilon}$

201 §2.3 #s 43, 45, 55-57, 59

43 cnt'd

2 $0 < |x-1| < \delta$

$-\delta < x-1 < \delta$

$1-\delta < x < 1+\delta$

$1-\delta = \frac{1}{1+\varepsilon}$ $\qquad$ $1+\delta = \frac{1}{1-\varepsilon}$

$-\delta = \frac{1}{1+\varepsilon} - 1$ $\qquad$ $\delta = \frac{1}{1-\varepsilon} - 1$

$\delta = 1 - \frac{1}{1+\varepsilon}$

Define $\boxed{\delta = \min\left\{1 - \frac{1}{1+\varepsilon}, \frac{1}{1-\varepsilon} - 1\right\}}$

45 $\lim_{x \to -3} \frac{x^2-9}{x+3} = -6$

1 $\left| \frac{x^2-9}{x+3} - (-6) \right| < \varepsilon$

$\left| \frac{(x-3)(x+3)}{x+3} + 6 \right| < \varepsilon$

$|x-3+6| < \varepsilon$

$|x+3| < \varepsilon$

$-\varepsilon < x+3 < \varepsilon$

$-3-\varepsilon < x < -3+\varepsilon$

2 $|x+3| < \delta$

$-\delta < x+3 < \delta$

$-3-\delta < x < \delta-3$

$-3-\delta = -3-\varepsilon$ $\qquad$ $\delta-3 = \varepsilon-3$

$\delta = \varepsilon$ $\qquad$ $\delta = \varepsilon$

$\boxed{\text{Let } \delta = \varepsilon}$

201 § 2.3 ~~#s~~ 55–57, 59

55 How much can you deviate from diameter $= 3.385$ in to obtain cross-sectional area of 9 in², if you must come w/in .01 in² of the 9 in².

$A = \pi \left(\frac{x}{2}\right)^2$, where $x =$ diameter, in inches

Want $\left|\frac{\pi x^2}{4} - 9\right| < .01 \implies$

$$-.01 < \frac{\pi x^2}{4} - 9 < .01$$

$$8.99 < \frac{\pi x^2}{4} < 9.01$$

$$4(8.99) < \pi x^2 < 4(9.01)$$

$$\frac{4(8.99)}{\pi} < x^2 < \frac{4(9.01)}{\pi}$$

$$\sqrt{\frac{4(8.99)}{\pi}} < |x| = x < \sqrt{\frac{4(9.01)}{\pi}} \quad (\text{since } x > 0)$$

oops! Find interval for x: simply

$$\boxed{\left(\sqrt{\frac{4(8.99)}{\pi}}, \sqrt{\frac{4(9.01)}{\pi}}\right) \approx (3.38702, 3.38326)}$$

Now $0 < |x - 3.385| < \delta \implies$

$$-\delta < x - 3.385 < \delta \implies$$

$$3.385 - \delta < x < 3.385 + \delta$$

$$3.385 - \delta = \sqrt{\frac{4(8.99)}{\pi}}$$

$$-\delta = \sqrt{\frac{4(8.99)}{\pi}} - 3.385$$

$$\delta = 3.385 - \sqrt{\frac{4(8.99)}{\pi}}$$

$$\approx .0017436533$$

$$3.385 + \delta = \sqrt{\frac{4(9.01)}{\pi}}$$

$$\delta = \sqrt{\frac{4(9.01)}{\pi}} - 3.385$$

$$\approx .0020176111$$

Min is!

201 § 2.3 #s 56, 57, 59

(56) $V = RI$ is Ohm's Law.

V = voltage = 120 = constant.

Find interval for R, given we need $I = 5 \pm .1$

$I = \frac{V}{R} = \frac{120}{R}$. want

$$\left| \frac{120}{R} - 5 \right| < .1$$

$$-.1 < \frac{120}{R} - 5 < .1$$

$$4.9 < \frac{120}{R} < 5.1$$

$$\frac{1}{5.1} < \frac{R}{120} < \frac{1}{4.9}$$

$$\frac{120}{5.1} < R < \frac{120}{4.9}$$

$$\boxed{R \in \left(\frac{120}{5.1}, \frac{120}{4.9} \right) \approx (23.5294, 24.4898)}$$

(57) $f(x) = \begin{cases} x, & x < 1 \\ x+1, & x > 1 \end{cases}$

(a) Let $\varepsilon = \frac{1}{2}$. Show $\not\exists\, \delta > 0 \ni$

$|f(x) - 2| < \frac{1}{2} = \varepsilon$ $\forall\ 0 < |x - 1| < \delta \implies$

Let $\delta > 0$. Then $x = 1 - \frac{\delta}{2}$ satisfies $0 < |x-1| < \delta$,

but $f(x) = f(1 - \frac{\delta}{2}) = 1 - \frac{\delta}{2} < 1$, and so

$|f(x) - 2| > 1$ ∎

201 §2.3 #s 59

57 cnt'd

(b) Show that $\lim_{x\to 1} f(x) \neq 1$.

Again, Let $\varepsilon = \frac{1}{2}$. For any $\delta > 0$, pick $x = 1 + \frac{\delta}{2}$.

Then $f(x) = f(1+\frac{\delta}{2}) = 1 + \frac{\delta}{2} + 1 = 2 + \frac{\delta}{2}$ and so

$|f(x) - 1| = 1 + \frac{\delta}{2} > \frac{1}{2}$

(c) Show that $\lim_{x\to 1} f(x) \neq 1.5$

Let $\varepsilon = \frac{1}{4}$. Let $\delta > 0$. Then $x = 1 + \frac{\delta}{2} \Rightarrow$

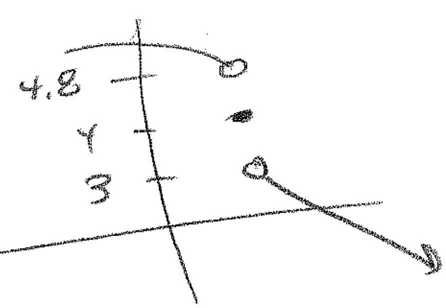
$f(x) = 2 + \frac{\delta}{2}$ and $|2 + \frac{\delta}{2} - 1.5| = |\frac{1}{2} + \frac{\delta}{2}| > \frac{1}{4}$

This'll be better after §2.4

59 — This'll be better after §2.4

(a) $\lim_{x\to 3} f(x) \neq 4$. Let $\varepsilon = \frac{1}{2}$. Then let $\delta > 0$. If $x = 1 + \frac{\delta}{2}$, we have $f(x) < 3$ and

$|3 - 4| = 1 > \frac{1}{2}$



(b) $\lim_{x\to 3} f(x) \neq 4.8$. Let $\varepsilon = 1$. Let $\delta > 0$, and $x = 3 + \frac{\delta}{2}$. Then $0 < |x - 3| < \delta$ and $|f(x) - 4.8| > 1.8$

(c) $\lim_{x\to 3} f(x) \neq 3$. Let $\varepsilon = 1$. Let $\delta > 0$ & $x = 3 - \frac{\delta}{2}$. Then $|x - 3| = \frac{\delta}{2} < \delta$ and $|f(x) - 3| > 1.8$