Section 11.11 #s 5, 6, 11, 12, 13, 14, plus either one of 33 or 36

Recall, Power series representation of $f(x)$ as a power series, via Taylor series expansion about

$x = a$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and the nth-degree Taylor polynomial of f at a.

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$ n^{th} partial sum.

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$\rightarrow f(x)$ as $n \rightarrow \infty$

$f(x) \approx T_n(x)$.

Einstein's theory of special relativity

the mass m of an object moving with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

New ton said $K = \frac{1}{2}mv^2$

where m_0 is the mass of the object when at rest and c is the speed of light.

The kinetic energy of the object is the difference between its total energy and its energy at rest:

$$K = mc^2 - m_0c^2$$

(a) Show that when v is very small compared with c, this expression for K agrees with classical Newtonian physics: $K = \frac{1}{2}m_0v^2$.

$e = mc^2$

$$K = mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} - m_0c^2 = m_0c^2\left[\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right]$$

$$x = -v^2/c^2 \qquad m_0c^2\left[\left(1 + \left(-\frac{v^2}{c^2}\right)\right)^{-\frac{1}{2}} - 1\right]$$

$$(1 + x)^{-1/2} = 1 - \tfrac{1}{2}x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}x^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}x^3 + \cdots$$

$$= 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3 + \cdots$$

$$K = m_0c^2\left[\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right) - 1\right]$$

$$= m_0c^2\left(\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right) = m_0\left[\frac{1}{2}v^2 + \frac{3}{8}\frac{v^4}{c^2} + \frac{5}{16}\frac{v^6}{c^4} + \cdots\right]$$

$c = 3 \times 10^8$ m/s

300,000,000 m/s

If v is much smaller than c,

very small, when v is small

$$K \approx m_0c^2\left(\frac{1}{2}\frac{v^2}{c^2}\right) = \tfrac{1}{2}m_0v^2$$

3–10 Find the Taylor polynomial $T_3(x)$ for the function f centered at the number a. Graph f and T_3 on the same screen.

5. $f(x) = \cos x, \quad a = \pi/2$

Use Wolfram Alpha!!!

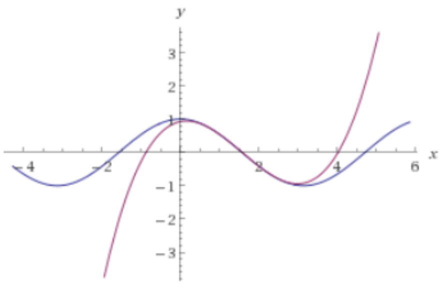
$f(x) = \cos x \qquad \cos\frac{\pi}{2} = 0$

$f'(x) = -\sin x \qquad -\sin\frac{\pi}{2} = -1$

$f''(x) = -\cos x \qquad -\cos\frac{\pi}{2} = 0$

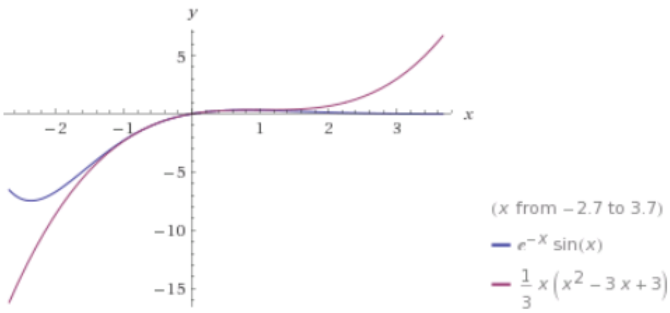
$f'''(x) = \sin x \qquad \sin\frac{\pi}{2} = 1$

$$T_3(x) = \frac{-1}{1!}\left(x-\frac{\pi}{2}\right)^1 + \frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3$$
$$= -\left(x-\frac{\pi}{2}\right) + \frac{1}{6}\left(x-\frac{\pi}{2}\right)^3$$



6. $f(x) = e^{-x} \sin x, \quad a = 0$

$T_3(x) = x - x^2 + \frac{x^3}{3}$



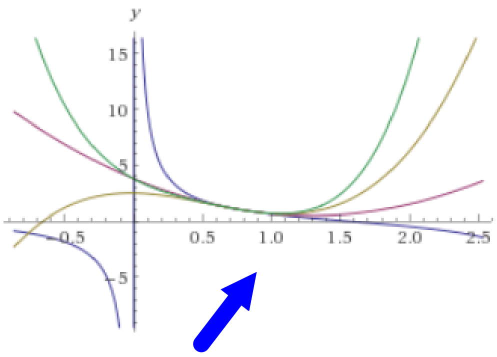
11–12 Use a computer algebra system to find the Taylor polynomials T_n centered at a for $n = 2, 3, 4, 5$. Then graph these polynomials and f on the same screen.

11. $f(x) = \cot x, \quad a = \pi/4$

Here's one with a restricted domain, fo' sho'.

$$1 - 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 - \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \frac{10}{3}\left(x - \frac{\pi}{4}\right)^4 - \frac{64}{15}\left(x - \frac{\pi}{4}\right)^5 + O\left(\left(x - \frac{\pi}{4}\right)^6\right)$$

I could only get the WolframAlpha site to handle up to degree 4 all on the same graph, and then the input got too long...



12. $f(x) = \sqrt[3]{1 + x^2}, \quad a = 0$

Degree 2, 3, 4 in plot is fine.

$$(1+x)^{\frac{1}{3}} = \sum_{n=0}^{\infty} \binom{k}{n} x^n = \binom{\frac{1}{3}}{0} x^0 + \binom{\frac{1}{3}}{1} x + \binom{\frac{1}{3}}{2} x^2 + \binom{\frac{1}{3}}{3} x^3 + \binom{\frac{1}{3}}{4} x^4 + \cdots$$

$k = \frac{1}{3}$

$$= 1 + \frac{\frac{1}{3}}{1!} x + \frac{(\frac{1}{3})(-\frac{2}{3})}{2!} x^2 + \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{3!} x^3 + \left(\frac{5}{81}\right)\left(\frac{-\frac{8}{3}}{4}\right) x^4$$

$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{\frac{10}{27}}{6} x^3 - \frac{10}{243} x^4 + \cdots -$$

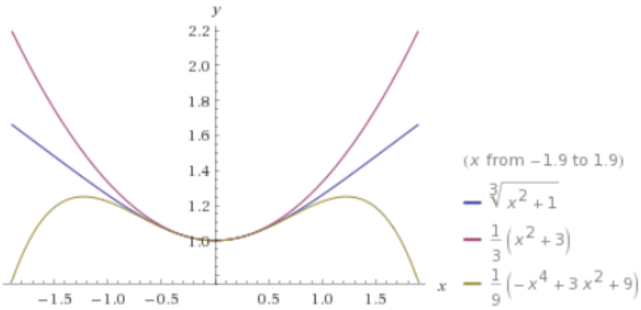
$$\frac{5}{81} \cdot \frac{\cancel{8}^{\,2}}{\cancel{12}_{\,3}}$$

$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3$$

Now, sub in x^2 for x:

$$1 + \frac{1}{3}x^2 - \frac{1}{9}x^4 + \frac{5}{81}x^6 - \frac{10}{243}x^8$$

$$\frac{\frac{10}{27}}{\frac{6}{1}} = \frac{10}{27} \cdot \frac{1}{6} = \frac{5}{81}$$



13–22

(a) Approximate f by a Taylor polynomial with degree n at the number a.

(b) Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval.

(c) Check your result in part (b) by graphing $|R_n(x)|$.

13. $f(x) = \sqrt{x}$, $a = 4$, $n = 2$, $4 \leqslant x \leqslant 4.2$

(a) $2 + \frac{1}{2^2}(x-4) - \frac{1}{2^6}(x-4)^2$

$= T_2(x)$

$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$

$2 + \frac{1}{2^2}(x-4) - \frac{1}{2^5}(x-4)^2 + \sum_{n=3}^{\infty} (-1)^{n-1}\left(\frac{3\cdot5\cdots(2n-3)}{n!\,2^{3n-1}}\right)$

$f(4) = 2$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ $\qquad f'(4) = \frac{1}{2}\frac{1}{2} = \frac{1}{2^2}$

$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$ $\qquad f''(4) = -\frac{1}{2^2}\cdot\frac{1}{2^3} = -\frac{1}{2^5}$

$f'''(x) = +\frac{3}{2^3}x^{-\frac{5}{2}}$ $\qquad f'''(4) = +\frac{3}{2^3}\frac{1}{2^5} = +\frac{3}{2^8}$

$f^{(4)}(x) = -\frac{3\cdot5}{2^4}x^{-\frac{7}{2}}$ $\qquad f^{(4)}(4) = -\frac{3\cdot5}{2^4}\cdot\frac{1}{2^7} = -\frac{15}{2^{11}}$

Taylor's Inequality

$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$

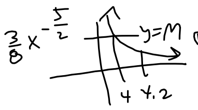
M is a bound on $f^{(n+1)}(x)$ on $I = [4, 4.2]$

$2n+1$
$2n+3$
$2n-7$

$f^{(n)}(x) = (-1)^{n-1}\frac{3\cdot5\cdots(2n-3)}{2^n}x^{-\frac{2n-1}{2}}$

$f^{(n)}($

$f^{(3)}(x)$ $\quad f^{(n+1)}(x) = \frac{3}{8}x^{-\frac{5}{2}}$



Decreasing

$|f^{(n+1)}(x)| \le |f^{(3)}(4)|$

$= \frac{3}{2^8} \equiv M$

(b) Find bound on $|f'''(x)|$

$I = [4, 4.2]$

$|R_2(x)| \le \frac{M}{3!}|x-4|^3$

$= \frac{3}{2^8}\cdot\frac{1}{3!}|x-4|^3$

$\le \frac{1}{2^9}(.2)^3 = \frac{1}{2^9}\left(\frac{2^3}{10^3}\right)$

$= \frac{1}{2^6 10^3} = \frac{1}{64000} \approx$

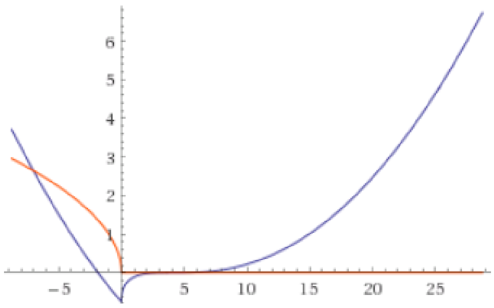
$\approx$ 0.00001562500000

(c) GRAPH $R_n(x)$! ?

CAKE

$f(x) = T_2(x) + R_2(x)$

$R_2(x) = f(x) - T_2(x)$



14. $f(x) = x^{-2}, \quad a = 1, \quad n = 2, \quad 0.9 \leqslant x \leqslant 1.1$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(a) $f(x) = x^{-2}$ $\qquad f(1) = 1$

$f'(x) = -2x^{-3}$ $\qquad f'(1) = -2$

$f''(x) = 6x^{-4} = 3!\,x^{-4}$ $\qquad f''(1) = 6$

$f^{(3)}(x) = -4!\,x^{-5}$ $\qquad f'''(1) = -24$

$f^{(4)}(x) = 5!\,x^{-6}$ $\qquad f^{(4)}(1) = 120$

$T_2(x) = 1 - \frac{2}{1!}(x-1)^1 + \frac{6}{2!}(x-1)^2$

$$\boxed{T_2(x) = 1 - 2(x-1) + 3(x-1)^2}$$

(b) $|R_2(x)| \leq \frac{M}{3!}|x-1|^3$

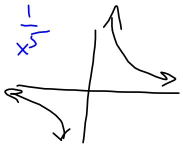
$M \geq |f^{(3)}(x)|$

$f^{(3)}(x) = -24x^{-5}$

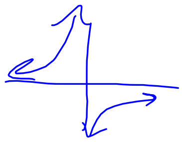
$|f^{(3)}|$ is decreasing

$\Rightarrow |f^{(3)}(x)| \leq |f^{(3)}(.9)| = \frac{24}{.9^5}$

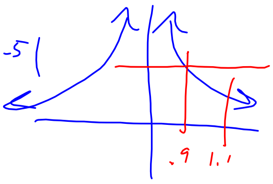
$\frac{1}{x^5}$



$-24x^{-5} = -\frac{24}{x^5}$



$|-24x^{-5}|$



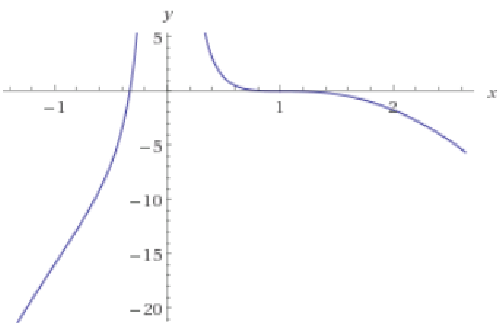
$|f^{(3)}(.9)| = M = \frac{24}{.9^5} \approx$ 40.64421074

$\Longrightarrow |R_2(x)| \leq \frac{\frac{24}{.9^5}}{3!}|x-1|^3 \leq \frac{24}{(.9^5)3!}(.1)^3 \approx$ 0.00006774035123

(c) $R_2(x) = f(x) - T_2(x)$

$= x^{-2} - (1 - 2(x-1) + 3(x-1)^2$

$= 1 - 2(x-1) + 3(x-1)^2$



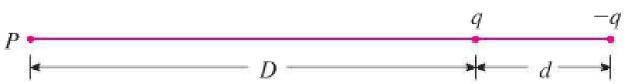
(x from −1.4 to 2.6)

25. Use Taylor's Inequality to determine the number of terms of the Maclaurin series for e^x that should be used to estimate $e^{0.1}$ to within 0.00001.

33. An electric dipole consists of two electric charges of equal magnitude and opposite sign. If the charges are q and $-q$ and are located at a distance d from each other, then the electric field E at the point P in the figure is

$$E = \frac{q}{D^2} - \frac{q}{(D + d)^2}$$

By expanding this expression for E as a series in powers of d/D, show that E is approximately proportional to $1/D^3$ when P is far away from the dipole.



36. A uniformly charged disk has radius R and surface charge density σ as in the figure. The electric potential V at a point P at a distance d along the perpendicular central axis of the disk is

$$V = 2\pi k_e \sigma \left(\sqrt{d^2 + R^2} - d\right)$$

where k_e is a constant (called Coulomb's constant). Show that

$$V \approx \frac{\pi k_e R^2 \sigma}{d} \qquad \text{for large } d$$