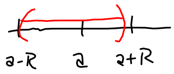
Section 11.10: Taylor Series #s 1, 2, 4 - 6, 8, 10, 11, 13, 18, 22, 26, 47, 54

Suppose a function has the power series representation

Has derivatives of all orders, we good!

$\boxed{1}$ $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$

$c_0 = f(a)$ $\qquad |x-a| < R$



ϕ converges possibly at $x = a \pm R$. Check manually.

$\boxed{2}$ $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$

$f'(a) = c_1$

$\boxed{3}$ $f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + \cdots$

$\frac{1}{1-x}$ trickerations

$f''(a) = 2c_2 \Rightarrow c_2 = \frac{f''(a)}{2}$

$\boxed{4}$ $f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x-a) + 3 \cdot 4 \cdot 5c_5(x-a)^2 + \cdots$

$f'''(a) = 2 \cdot 3\, c_3 \Rightarrow c_3 = \frac{f'''(a)}{3!}$

$f^{(n)}(a) = 2 \cdot 3 \cdot 4 \cdot \cdots \cdot nc_n = n!c_n$

$$c_n = \frac{f^{(n)}(a)}{n!}$$ SWEET!

5 Theorem If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$f^{(n)}(a) = n!\, c_n$

$\Rightarrow \boxed{c_n = \frac{f^{(n)}(a)}{n!}}$ → How we find a power series' representation, one term at a time.

6 $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

f''

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

The series in Equation 6 is called the **Taylor series of the function *f* at *a*** (or **about *a*** or **centered at *a***).

a is the center of the interval of convergence.

For the special case *a* = 0 the Taylor series becomes

7 $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

a

This case arises frequently enough that it is given the special name **Maclaurin series**.

Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

$0! = 1$

$f(x) = e^x \quad f(0) = e^0 = 1 \quad T_0 = 1 = \frac{f(0)}{0!} x^0 = 1$

$f'(x) = e^x \quad f'(0) = e^0 = 1 \quad T_1 = 1 + \frac{1}{1!} x^1$

$f''(x) = e^x \quad f''(0) = e^0 = 1 \quad T_2 = \frac{1}{0!} x^0 + \frac{1}{1!} x^1 + \frac{1}{2!} x^2$

$\vdots$

$f^{(n)}(x) = e^x \quad f^{(n)}(0) = 1 \quad T_n = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \dots + \frac{1}{n!} x^n$

$= \sum_{k=0}^{n} \frac{1}{k!} x^k$

$\xrightarrow{n \to \infty} \sum_{k=0}^{\infty} \frac{1}{k!} x^k = e^x$

Wolfram Alpha

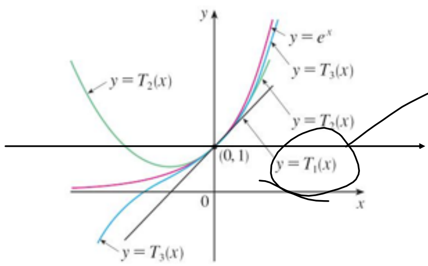
$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Notice that T_n is a polynomial of degree n called the ***n*th-degree Taylor polynomial of *f* at *a***.

$T_0(x) = 1$

$T_1(x) = 1 + x \qquad T_2(x) = 1 + x + \frac{x^2}{2!} \qquad T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$



$y = T_0 = 1$

Is the tangent line @ $x = a$

$= f'(a)(x-a) + f(a)$

$= f(a) + f'(a)(x-a)$

$= \frac{f(a)}{0!}(x-a)^0 + \frac{f'(a)}{1!}(x-a)^1$

Figure 1

As n increases, $T_n(x)$ appears to approach e^x in Figure 1. This suggests that e^x is equal to the sum of its Taylor series.

In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n \to \infty} T_n(x)$$

If we let

The n-tail $R_n = c_{n+1}x^{n+1} + c_{n+2}x^{n+2} + \cdots$ = Error

$\boxed{R_n(x) = f(x) - T_n(x)}$ so that $f(x) = T_n(x) + R_n(x)$

then $R_n(x)$ is called the **remainder** of the Taylor series. If we can somehow show that $\lim_{n \to \infty} R_n(x) = 0$, then it follows that

$$\lim_{n \to \infty} T_n(x) = \lim_{n \to \infty} [f(x) - R_n(x)] = f(x) - \lim_{n \to \infty} R_n(x) = f(x)$$

8 Theorem If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and

$$\lim_{n \to \infty} R_n(x) = 0$$

= Radius of convergence

for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $|x - a| < R$.

9 Taylor's Inequality If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

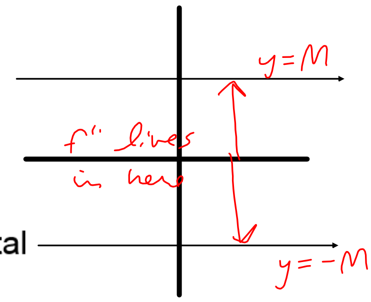
$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1} \qquad \text{for } |x - a| \leq d$$

Recipe

Get a bound on $|f^{(n+1)}|$ to get a bound on the error.

To see why this is true for $n = 1$, we assume that $|f''(x)| \le M$. In particular, we have $f''(x) \le M$, so for $a \le x \le a + d$ we have

$$f'(x) - f'(a) = \int_a^x f''(t)\,dt \leqslant \int_a^x M\,dt = M(x-a)$$



An antiderivative of f'' is f', so by Part 2 of the Fundamental Theorem of Calculus, we have

$$f'(x) - f'(a) \le M(x - a) \quad \text{or} \quad f'(x) \le f'(a) + M(x - a)$$

Thus
$$\int_a^x f'(t)\,dt \leqslant \int_a^x [f'(a) + M(t-a)]\,dt$$

$$f(x) - f(a) \leqslant f'(a)(x-a) + M\frac{(x-a)^2}{2}$$

$$f(x) - f(a) - f'(a)(x-a) \leqslant \frac{M}{2}(x-a)^2$$

$$f(x) - T_1(x) = R_1(x) \le \frac{M}{2!}(x-a)^2 \le \frac{M}{2!}|x-a|^2$$

But $R_1(x) = f(x) - T_1(x) = f(x) - f(a) - f'(a)(x - a)$. So

$$R_1(x) \leqslant \frac{M}{2}(x-a)^2$$

A similar argument, using $f''(x) \ge -M$, shows that

$$R_1(x) \geqslant -\frac{M}{2}(x-a)^2$$

So
$$|R_1(x)| \leqslant \frac{M}{2}|x-a|^2$$

Need this, for FORMAL proof, but the stuff above this basically nails the idea, so we good.

Although we have assumed that $x > a$, similar calculations show that this inequality is also true for $x < a$. $|x-a| = a-x$

This proves Taylor's Inequality for the case where $n = 1$. The result for any n is proved in a similar way by integrating $n + 1$ times.

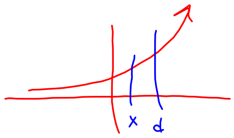
In applying Theorems 8 and 9 it is often helpful to make use of the following fact.

$x^n \leq n! \leq n^n$ eventually.

10 $$\lim_{n \to \infty} \frac{x^n}{n!} = 0 \qquad \text{for every real number } x$$ Handy

This is true because we know from Example 1 that the series $\Sigma\, x^n/n!$ converges for all x and so its nth term approaches 0.

Prove that e^x is equal to the sum of its Maclaurin series.



Solution:

If $f(x) = e^x$, then $f^{(n+1)}(x) = e^x$ for all n. If d is any positive number and $|x| \le d$, then $|f^{(n+1)}(x)| = e^x \le e^d$, because e^x is increasing

So Taylor's Inequality, with $a = 0$ and $M = e^d$, says that

$$|R_n(x)| \leqslant \underbrace{\frac{e^d}{(n+1)!}|x|^{n+1}}_{\xrightarrow{n \to \infty} 0} \qquad \text{for } |x| \leqslant d$$

$|x|^n$ is overpowered by $n!$

i.e., $S_n = T_n \xrightarrow{n \to \infty} e^x$!!!

$$\boxed{12} \quad e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Recall Binomial Coefficient "k choose n" from College Algebra & The Binomial Theorem (Pascal's Triangle)

k is positive integer
$k \geq n$

$$\binom{k}{n} = \frac{k!}{n!(n-k)!}$$

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = \frac{5\cdot4\cdot\cancel{3\cdot2}}{2\cdot\cancel{3\cdot2}} = \frac{5\cdot4}{2!}$$

= 10 ways to pick 2 fingers out of 5 fingers

{1,2}, {1,3}, {1,4}, {1,5},
{2,3}, {2,4}, {2,5},
{3,4}, {3,5},
{4,5}

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \\ \binom{4}{0}&&\binom{4}{1}&&\binom{4}{2}&&\binom{4}{3}&&\binom{4}{4} \end{array}$$

Binomial Theorem

$$\sum_{n=0}^{k}\binom{k}{n}x^k = (1+x)^k = \binom{k}{0}x^0 + \binom{k}{1}x^1 + \binom{k}{2}x^2 + \cdots + \binom{k}{k}x^k$$

$$(1+x)^4 = 1x^0 + 4x + 6x^2 + 4x^3 + 1x^4$$

Now, we extend to k = any real number & we look @ $(1+x)^k$

Find the Maclaurin series for $f(x) = (1 + x)^k$, where k is any real number.

Arranging our work in columns, we have

$$
\begin{aligned}
f(x) &= (1 + x)^k & f(0) &= 1 \\
f'(x) &= k(1 + x)^{k-1} & f'(0) &= k \\
f''(x) &= k(k-1)(1 + x)^{k-2} & f''(0) &= k(k-1) \\
f'''(x) &= k(k-1)(k-2)(1 + x)^{k-3} & f'''(0) &= k(k-1)(k-2) \\
&\vdots & &\vdots \\
f^{(n)}(x) &= k(k-1)\cdots(k-n+1)(1+x)^{k-n} & f^{(n)}(0) &= k(k-1)\cdots(k-n+1)
\end{aligned}
$$

Therefore the Maclaurin series of $f(x) = (1 + x)^k$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n = (1+x)^k$$

$\binom{k}{n}$

looks a lot like

$$\frac{k!}{n!\,(k-n)!} = \frac{k\cdot(k-1)(k-2)\cdots(k-n+1)\cancel{(k-n)!}}{n!\,\cancel{(k-n)!}}$$

This series is called the **binomial series**.

If its nth term is a_n, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k-1)\cdots(k-n+1)x^n}\right|$$

$$= \frac{|k-n|}{n+1}|x| = \frac{\left|1 - \frac{k}{n}\right|}{1 + \frac{1}{n}}|x| \to |x| \quad \text{as } n \to \infty$$

Thus, by the Ratio Test, the binomial series converges if $|x| < 1$ and diverges if $|x| > 1$.

Leaves $x = \pm 1$ up for grabs

The traditional notation for the coefficients in the binomial series is

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

$(k-n)!$
$(k-n)!$

and these numbers are called the **binomial coefficients**. The following theorem states that $(1 + x)^k$ is equal to the sum of its Maclaurin series.

It is possible to prove this by showing that the remainder term $R_n(x)$ approaches 0, but that turns out to be quite difficult.

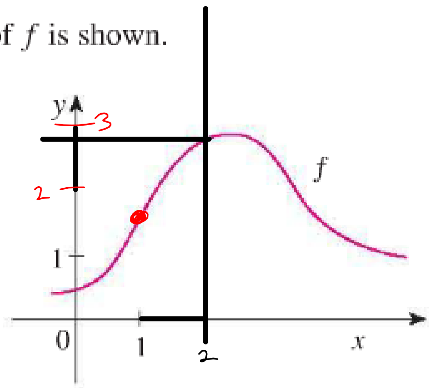
17 The Binomial Series If k is any real number and $|x| < 1$, then

$$(1+x)^k = \boxed{\sum_{n=0}^{\infty} \binom{k}{n} x^n} = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

1. If $f(x) = \sum_{n=0}^{\infty} b_n(x-5)^n$ for all x, write a formula for b_8.

$$b_8 = \frac{f^{(8)}(5)}{8!}$$

2. The graph of f is shown.



(a) Explain why the series

$$1.6 - 0.8(x-1) + 0.4(x-1)^2 - 0.1(x-1)^3 + \cdots$$

is *not* the Taylor series of f centered at 1.

(b) Explain why the series

$$2.8 + 0.5(x-2) + 1.5(x-2)^2 - 0.1(x-2)^3 + \cdots$$

is *not* the Taylor series of f centered at 2.

(a) $f'(1) = .8$ is too small

(b) $f''(2) \neq 2 \cdot 1.5 > 0$ but f is concave down @ $x = 2$.

4. Find the Taylor series for f centered at 4 if

$$f^{(n)}(4) = \frac{(-1)^n n!}{3^n(n+1)}$$

What is the radius of convergence of the Taylor series?

$$T = \sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!}(x-4)^n = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!\,3^n(n+1)}(x-4)^n$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)\cdot 3^n}(x-4)^n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-4)^{n+1}}{(n+2)3^{n+1}} \cdot \frac{(n+1)3^n}{(x-4)^n}\right|$$

$$\xrightarrow{n\to\infty} \frac{|x-4|}{3} \overset{\text{want}}{<} 1$$

$$|x-4| < \boxed{3 = R}$$

5–12 Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.

5. $f(x) = (1 - x)^{-2}$

$f(0) = 1 = 1!$

$f'(0) = 2 = 2!$

$f''(0) = 6 = 3!$

$f'''(0) = 4!$

$\vdots$

$f^{(n)}(0) = (n+1)!$

$f'(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$

$f''(x) = -6(1-x)^{-4}(-1) = 6(1-x)^{-4}$

$f'''(x) = -24(1-x)^{-5}(-1) = 24(1-x)^{-5}$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!} x^n$$

$$= \sum_{n=0}^{\infty} (n+1) x^n = (1-x)^{-2}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+2)x^{n+1}}{(n+1)x^n} \right| \xrightarrow{n \to \infty} |x| < \boxed{1 = R}$$

6. $f(x) = \ln(1 + x)$

§11.9 see $\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \dots$

$= \frac{d}{dx}\left[\ln(1+x)\right]$

$\ln(1+x) = \int (1 - x + x^2 - x^3 + \dots)\,dx$

$= C + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$

$\xi \ln(1+0) = C$

$\ln(1) = 0 = C$

$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$

$f(x) = \ln(1+x)$	$f(0) = 0$	
$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$	$f'(0) = 1$ *	$\frac{1}{1!}x^1$
$f''(x) = -(1+x)^{-2}$	$f''(0) = -1$	$\frac{-1}{2!}x^2$
$f'''(x) = 2(1+x)^{-3} = 2!(1+x)^{-3}$	$f'''(0) = 2$	$\frac{2!}{3!}x^3$
$f^{(4)}(x) = -3!(1+x)^{-3}$	$f^{(4)}(0) = -3!$	$-\frac{3!}{4!}x^4$
	$\vdots$	$\vdots$
	$f^{(n)}(0) = (-1)^{n+1}(n-1)!$	$\frac{(-1)^{n+1}(n-1)!}{n!}x^n$

$\left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right|$

$\xrightarrow{n\to\infty} |x| < \boxed{1 = R}$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!} x^n$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$ ✓

$\boxed{M1}$ Sub $-2x$ for x in e^x: $1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$$e^{-2x} = 1 + (-2x) + \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3!} + \cdots$$

$$= 1 - 2x + \frac{2^2x^2}{2} - \frac{2^3x^3}{3!} + \frac{2^4x^4}{4!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n 2^n \frac{x^n}{n!}$$

8. $f(x) = e^{-2x}$

$f'(x) = -2e^{-2x}$ $\quad f'(0) = -2$

$f''(x) = (-2)^2 e^{-2x}$ $\quad f''(0) = 2^2$

$f'''(x) = (-2)^3 e^{-2x}$ $\quad f'''(0) = -2^3$

$\vdots$

$f^{(n)}(x) = (-2)^n e^{-2x} = (-1)^n 2^n e^{-2x}$ $\quad f^{(n)}(0) = (-2)^n = (-1)^n (2)^n$

$$a_n = \frac{(-1)^n 2^n}{n!} x^n$$

See? Same terms

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}x^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n x^n}\right| = \left|\frac{2x}{n+1}\right| \xrightarrow{n\to\infty} 0 < 1$$

So

No x, here, so "< 1" regardless of x!

$$\boxed{R = \infty}$$

10. $f(x) = x\cos x$

$f'(x) = \cos x + x(-\sin x)$
$= \cos x - x\sin x$ $\qquad f'(0) = 1 \qquad \frac{1}{1!}x^1$

$f''(x) = -\sin x - [1\sin x + x\cos x]$ $\qquad f''(0) = 0$
$= -\sin x - \sin x - x\cos x$
$= -2\sin x - x\cos x$

$f'''(x) = -2\cos x - \cos x + x\sin x$ $\qquad f'''(0) = -3 \qquad \frac{-3}{3!}x^3$
$= -3\cos x + x\sin x$

$f^{(4)}(x) = +3\sin x + \sin x + x\cos x$ $\qquad f^{(4)}(0) = 0$
$= 4\sin x + x\cos x$

$f^{(5)}(x) = 5\cos x - x\sin x$ $\qquad f^{(5)}(0) = 5 \qquad \frac{5}{5!}x^5$

$\vdots$

$$f^{(n)}(x) = \begin{cases} 0 & \text{if } \quad 2k+1 \quad n = 2k-1,\ k = 1, 2, 3, \dots \\ -1 & \text{if } \quad n = 2k,\ k \end{cases}$$

Too hard! Just logic out the $\sum$ from the 1st few terms

$x - \frac{1}{2!}x^3 + \frac{1}{4!}x^5 - \dots$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-2)!} =$$

$n = k+1$

$$\sum_{k=0}^{\infty} (-1)^{k+1+1} \frac{x^{2(k+1)-1}}{(2(k+1)-2)!} =$$

OR JUST k

$$= \sum_{k=0}^{\infty} (-1)^{k+2} \frac{x^{2k+1}}{(2k)!}$$

$$\boxed{\sum_{k=0}^{\infty} (-1)^{k} \frac{x^{2k+1}}{(2k)!}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2(n+1)+1}}{(2(n+1))!} \cdot \frac{(2n)!}{x^{2n+1}}\right| = \left|\frac{x^{2n+3}}{(2n+2)(2n+1)\,x^{2n+1}}\right| = \frac{|x|^3}{(2n+2)(2n+1)}$$

$\xrightarrow{n\to\infty} 0 < 1$ regardless of x

$$\frac{(2n)!}{(2n+2)!} = \frac{(2n)!}{(2n+2)(2n+1)(2n)!}$$

$\boxed{R = \infty}$

11. $f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$, btw. $\frac{e^x + e^{-x}}{2} = \cosh x$

$f'(x) = \cosh x$ $\quad f'(0) = 1$ $\quad \frac{1}{1!}x^1$

$f''(x) = \sinh x$ $\quad f''(0) = 0$ $\quad 0$

$f'''(x) = \cosh x$ $\quad f'''(0) = 1$ $\quad \frac{1}{3!}x^3$

$\vdots$ $\quad \frac{1}{5!}x^5$

$f^{(n)}(x)$

$$\boxed{\sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2(n+1)-1}}{(2(n+1)-1)!} \cdot \frac{(2n-1)!}{x^{2n-1}}\right|$$

$$= \left|\frac{x^2}{(2n+1)(2n)}\right| \xrightarrow{n \to \infty} 0 \quad \forall x$$

$$\boxed{R = \infty}$$

13–20 Find the Taylor series for $f(x)$ centered at the given value of a. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.

13. $f(x) = x^4 - 3x^2 + 1, \quad a = 1$

$f(1) = 1 - 3 + 1 = -1$ $\qquad \frac{-1}{0!}(x-1)^0 = -1$

$f'(x) = 4x^3 - 6x$ $\qquad f'(1) = 4 - 6 = -2$ $\qquad \frac{-2}{1!}(x-1)^1 = -2(x-1)$

$f''(x) = 12x^2 - 6$ $\qquad f''(1) = 12 - 6 = 6$ $\qquad \frac{6}{2!}(x-1)^2 = 3(x-1)^2$

$f'''(x) = 24x$ $\qquad f'''(1) = 24$ $\qquad \frac{24}{3!}(x-1)^3 = 4(x-1)^3$

$f^{(4)}(x) = 24$ $\qquad f^{(4)}(1) = 24$ $\qquad \frac{24}{4!}(x-1)^4 = (x-1)^4$

$f^{(5)}(x) = 0 = f^{(n)}(x) \quad \forall\, n \geq 5$ $\qquad 0$ $\qquad f(x$

$f(x) = -1 - 2(x-1) + 3(x-1)^2 + 4(x-1)^3 + (x-1)^4$

15. $f(x) = \ln x, \quad a = 2$

$\ln 2$

$f'(x) = \frac{1}{x} = x^{-1}$ $\quad f'(2) = \frac{1}{2}$

$f''(x) = -\frac{1}{x^2} = -x^{-2}$ $\quad f''(2) = -2^{-2} = -\frac{1}{2^2}$

$f'''(x) = 2x^{-3}$ $\quad f'''(2) = \frac{2}{2^3}$

$f^{(4)}(x) = -3 \cdot 2x^{-4}$ $\quad f^{(4)} = -\frac{3!}{2^4}$

$f^{(5)}(x) = 4 \cdot 3 \cdot 2 x^{-5} = 4! x^{-5} = \frac{4!}{2^5}$

$\vdots$

$f^{(n)}(x) = (n-1)! \, x^{-n}$ $\quad f^{(n)}(2) = (n-1)! \, 2^{-n} = \frac{(n-1)!}{2^n}$ $\quad \frac{(n-1)! \, x^n}{2^n \cdot n!}$

$\ln 2$

Build Pattern from down here & tack on $\ln(2)$

$\frac{1}{2} \frac{x}{1!}$

$-\frac{1}{2^2} \frac{x^2}{2!}$

$\frac{2}{2^3} \frac{x^3}{3!}$

$-\frac{3!}{2^4} \frac{x^4}{4!}$

$\frac{4!}{2^5} \frac{x^5}{5!}$

$\vdots$

No! Expanding about $a = 2$!

$$\sum_{n=1}^{\infty} \frac{(n-1)! \, (-1)^{n-1} x^n}{2^n n!}$$

$$= \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n \, 2^n}}$$

oopsie!

Not quite

Yes!

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1) 2^{n+1}} \cdot \frac{n 2^n}{x^n} \right|$$

$$\xrightarrow{n \to \infty} \frac{|x|}{2} < 1$$

$$|x| < \boxed{2 = R}$$

Need $\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x-2)^n}{n 2^n} \right) + \ln 2$

left this out, too!

18. $f(x) = \sin x, \quad a = \pi/2$ $\quad f(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$ $\quad \frac{1}{0!}(x-\frac{\pi}{2})^0 = 1$ $\quad n=0$ $\quad \frac{1}{0!}(x-\frac{\pi}{2})^0 =$

$f'(x) = \cos x$ $\quad f'(\frac{\pi}{2}) = \cos\frac{\pi}{2} = 0$ $\quad 0$

$f''(x) = -\sin x$ $\quad -1$ $\quad \frac{-1}{2!}(x-\frac{\pi}{2})^2$ $\quad n=1$ $\quad -\frac{1}{2!}(x-\frac{\pi}{2})^{2(1)}$

$f'''(x) = -\cos x$ $\quad 0$

$f^{(4)}(x) = \sin x$ $\quad 1$ $\quad \frac{1}{4!}(x-\frac{\pi}{2})^4$ $\quad n=2$ $\quad \frac{1}{4!}(x-\frac{\pi}{2})^{2(2)}$

$f^{(5)}(x) = \cos x$ $\quad 0$

$f^{(6)}(x) = -\sin x$ $\quad -1$ $\quad -\frac{1}{6!}(x-\frac{\pi}{2})^6$ $\quad n=3$

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n (x-\frac{\pi}{2})^{2n}}{(2n)!}}$$

$$\boxed{R = \infty}$$

22. Prove that the series obtained in Exercise 18 represents sin x for all x.

$f^{(n)}(x) = \pm\cos x$ OR $\pm\sin x$
depending on
n = even/odd

$|R_n| = \sum_{k=n+1}^{\infty} a_k$

$|f^{(n)}(x)| \le 1 \equiv M$

9 Taylor's Inequality If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \le d$$

$\forall$ f on $|x-a| \le d$

$|R_n(x)| \le \frac{1}{(n+1)!}|x-a|^{n+1} \le \frac{1}{(n+1)!}d$

$\xrightarrow{n\to\infty} 0$ for any x

proves it.

25–28 Use the binomial series to expand the function as a power series. State the radius of convergence.

17 **The Binomial Series** If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

26. $\sqrt[3]{8+x} = (8+x)^{\frac{1}{3}} = \left(8\left(1+\frac{x}{8}\right)\right)^{\frac{1}{3}}$

$= 8^{\frac{1}{3}}\left(1+\frac{x}{8}\right)^{\frac{1}{3}} = 2\left(1+\frac{x}{8}\right)^{\frac{1}{3}} = 2\sum_{n=0}^{\infty}\binom{\frac{1}{3}}{n}\left(\frac{x}{8}\right)^n = 1 + \frac{1}{3}\left(\frac{x}{8}\right) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}\left(\frac{x}{8}\right)^2$

$+ \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(\frac{x}{8}\right)^3 + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{4!}\left(\frac{x}{8}\right)^4$

$n=4$ $\quad 2\cdot5\cdot8 = 2\cdot5\cdots(3(n-1)-1)$

$= 2\cdot5\cdots3n-4$

$+ \quad (-1)^{n+1}\frac{(2)(5)(8)\cdots(3n-4)}{3^n\, n!}\frac{x^n}{8^n}$

$$2\left[1 + \frac{x}{24} + \sum_{n=2}^{\infty}(-1)^{n+1}\frac{(2)(5)\cdots(3n-4)\;x^n}{24^n\, n!}\right]$$

$= 2 + \frac{x}{12} + 2\sum$

28. $(1-x)^{2/3}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2\cdot5\cdots(3(n+1)-4)\;x^{n+1}}{24^{n+1}(n+1)!}\cdot\frac{24^n\,(n)!}{2\cdot5\cdots(3n-4)\;x^n}\right|$$

$\frac{(3n-1)}{24(n+1)}|x| \xrightarrow{n\to\infty} \frac{1}{8}|x| < 1$

$|x| < \boxed{8 = R}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

47–50 Evaluate the indefinite integral as an infinite series.

47. $\int x\cos(x^3)\,dx$ **48.** $\int \frac{e^x - 1}{x}\,dx$

$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

$$\Rightarrow \quad x\cos(x^3) = x\sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n}}{(2n)!} = x\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(2n)!}$$

& so, $$\int x\cos(x^3)\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(2n)!}\,dx$$

$$= \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!} \right) + C$$

51–54 Use series to approximate the definite integral to within the indicated accuracy.

54. $\int_0^{0.5} x^2 e^{-x^2}\,dx \quad (|\text{error}| < 0.001)$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$(-x^2)^n = ((-1)x^2)^n = (-1)^n x^{2n}$$

$$\int_0^{0.5} x^2 e^{-x^2} = \int_0^{.5} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,dx$$

$$= \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,n!}\right]_0^{.5}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)^{2n+1}}{(2n+1)\,n!}$$

So to be w/in .001, I need $|R_n| \le .001$. It's alternating, so

$$|R_n| \le |b_{n+1}| = \frac{\left(\frac{1}{2}\right)^{2n+1}}{(2n+1)\,n!} \le .001$$

$$|b_{n+1}| = \frac{\left(\frac{1}{2}\right)^{2n+1}}{(2n+1)\,n!} \le .001$$

n	
1	0.041667
2	0.003125
3	0.000186
4	9.04E-06
5	3.7E-07

< .001, so we just 1st 2 terms

ADD EM

0.044791667

Does the job!

53. $\int_0^{0.4} \sqrt{1+x^4}\,dx \quad (|\text{error}| < 5 \times 10^{-6})$