11.7 #s 1 - 24, 33, 34

11.7 Strategy for Testing Series

1. If the series is of the form $\Sigma\ 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leq 1$.

p-series

2. If the series has the form or $\Sigma\ ar^{n-1}$ or $\Sigma\ ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geq 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.

geometric

3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series.

Direct Comparison

Limit Comparison

$\frac{1}{n+1} < \frac{1}{n}$

$\frac{1}{n+700} \xrightarrow{n \to \infty} \frac{1}{n}$

I think Limit Comparison is stronger than that.

The comparison tests apply only to series with positive terms, but if Σa_n has some negative terms, then we terms, then we can apply the Comparison Test to $\Sigma\ |a_n|$ and test for absolute convergence.

$\lim_{n \to \infty} \left| \frac{a_n}{b_n} \right| = \text{constant}$

If $= 0$, then Σa_n is very well-behaved, indeed.

4. If you can see at a glance that $\lim_{n \to \infty} a_n \neq 0$, then the Test for Divergence should be used.

$\Sigma \frac{n^5}{n^6 - 11} \leftrightarrow \Sigma \frac{1}{n}$

$\Sigma (-1)^n \frac{n^5}{n^6 - 11} \leftrightarrow$

$\Sigma (-1)^n \frac{1}{n}$

5. If the series is of the form $\Sigma(-1)^{n-1}b_n$ or $\Sigma(-1)^n b_n$, then the Alternating Series Test is an obvious possibility.

6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.

7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.

8. If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

In the following examples we don't work out all the details but simply indicate which tests should be used.

$$\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$$

$a_n \longrightarrow \frac{1}{2} \neq 0$

Diverge

$$\sum_{k=1}^{\infty} \frac{2^k}{k!}$$

$\frac{2^{k+1}}{(k+1)!} \cdot \frac{k!}{2^k} = \frac{2}{k+1} \xrightarrow{k\to\infty} 0$

Conv. Abs.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$$

Intuit

$\frac{n^{3/2}}{3n^3} = \frac{1}{3n^{3/2}}$

converges Abs.

To show concretely, Limit Comparison to $\sum \frac{1}{3n^{3/2}}$

$$\sum_{n=1}^{\infty} \frac{1}{2+3^n}$$

$a_n = \frac{1}{3^n+2} < \frac{1}{3^n}$

Conv. Abs.

Compare to $\sum \frac{1}{3^n} = \sum \left(\frac{1}{3}\right)^n$

Geometric, $|r| = \frac{1}{3} < 1$

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$$

intuit $\sum \frac{1}{n}$

Converges Conditionally.

Limit Compare to $\sum \frac{1}{n}$

1–38 Test the series for convergence or divergence.

1. $\sum_{n=1}^{\infty} \frac{1}{n + 3^n}$

$a_n = \frac{1}{n+3^n} < b_n = \frac{1}{3^n}$

& $\sum b_n$ conv. abs

Comparison Test

2. $\sum_{n=1}^{\infty} \frac{(2n + 1)^n}{n^{2n}}$

$a_n = \frac{(2n+1)^n}{(n^2)^n} = \left(\frac{2n+1}{n^2}\right)^n$

Root Test

$\sqrt[n]{|a_n|} = \frac{2n+1}{n^2} \xrightarrow{n\to\infty} 0$

Conv. Abs

3. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$

$\frac{n}{n+2} \xrightarrow{n\to\infty} 1 \neq 0$

Fails Test for Divergence

$\Rightarrow$ Diverges

4. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+2}$

Converges Conditionally

Limit Comparison to $\sum (-1)^n \frac{1}{n}$

$\boxed{\text{Limit Comparison}}$ works for alternating series.

Strongest weopon.

$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| =$ Constant $\Rightarrow$ Both series have same convergence properties.

5. $\sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{(-5)^n}$

$|a_n| = \left|\frac{n^2 \cdot 2^n \cdot 2^{-1}}{(-5)^n}\right|$

$= n^2 \cdot \frac{1}{2} \cdot \left(\frac{2}{5}\right)^n$

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2 (2/5)^{n+1}}{n^2 \cdot (2/5)^n} =$

$\frac{(n+1)^2}{n^2} \cdot \frac{2}{5} \xrightarrow{n\to\infty} \frac{2}{5} < 1$

Conv. Abs

6. $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ Diverges

Limit comparison to

$\sum \frac{1}{2n}$

$= \sum \frac{1}{2} \cdot \frac{1}{n}$

$= \frac{1}{2} \sum \frac{1}{n}$

$\frac{\frac{1}{2n+1}}{\frac{1}{2n}} = \frac{2n}{2n+1} \xrightarrow{n\to\infty} 1$

So both diverge, since $\sum \frac{1}{2n}$ diverges

$\sum \frac{n^2 2^{n-1}}{(-5)^n}$

$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{n^2 2^n \cdot \frac{1}{2}}{5^n}} = \sqrt[n]{n^2 \left(\frac{2}{5}\right)^n \cdot \frac{1}{2}}$

$= (n^2)^{\frac{1}{n}} \left(\left(\frac{2}{5}\right)^n\right)^{\frac{1}{n}} \left(\frac{1}{2}\right)^{\frac{1}{n}}$

$r^n \xrightarrow{n\to\infty} 1$ if $0 < r < 1$

$= \left(n^{\frac{1}{n}}\right)^2 \left(\frac{2}{5}\right)\left(\frac{1}{2}\right) \xrightarrow{n\to\infty}$

$= (1^2)\left(\frac{2}{5}\right)(1) = \frac{2}{5} < 1$

conv. abs.

11. $\sum_{n=1}^{\infty}\left(\frac{1}{n^3}+\frac{1}{3^n}\right)$

$p=3$-series

$r=\frac{1}{3}$ geometric

Both converge Abs, separately so

Abs Conv.

12. $\sum_{k=1}^{\infty}\frac{1}{k\sqrt{k^2+1}}$

Limit Compare to $\sum\frac{1}{k^2}$ OR

Direct compare!

$$\frac{1}{k\sqrt{k^2+1}} < \frac{1}{k\sqrt{k^2}} = \frac{1}{k|k|}$$

$$= \frac{1}{k^2}, \text{ since } k>0$$

13. $\sum_{n=1}^{\infty}\frac{3^n n^2}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}(n+1)^2}{(n+1)!}\cdot\frac{n!}{3^n n^2}$$

$$= \frac{3^{n+1}}{3^n}\cdot\frac{(n+1)^2}{n^2}\cdot\frac{n!}{(n+1)!}$$

$$\xrightarrow{n\to\infty} 3\cdot 1\cdot\frac{1}{n+1} \xrightarrow{n\to\infty} 0$$

Conv. Abs.

14. $\sum_{n=1}^{\infty}\frac{\sin 2n}{1+2^n}$

Comparison to $\sum\frac{1}{2^n}$

conv. Abs.

$$\left|\frac{\sin(2n)}{1+2^n}\right| < \frac{1}{2^n}$$

$$=\left(\tfrac{1}{2}\right)^n$$

15. $\sum_{k=1}^{\infty} \frac{2^{k-1}3^{k+1}}{k^k}$

$2^k \cdot 3^k = 6^k$

kills others

Basically fastest-growing thing we'll have!

$\frac{2^{-1} \cdot 2^k \cdot 3^k \cdot 3^1}{k^k}$

$= \frac{3}{2} \cdot \frac{6^k}{k^k} =$

$\sqrt[n]{|a_n|} = \sqrt[k]{\frac{3}{2}\left(\frac{6}{k}\right)^k} = \left(\sqrt[k]{\frac{3}{2}}\right)\left(\frac{6}{k}\right) \xrightarrow{k \to \infty} 1 \cdot 0 = 0$

$\boxed{\text{Conv. abs.}}$

16. $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

Limit compare to $\sum \frac{1}{n}$

$\frac{\frac{n^2+1}{n^3+1}}{\frac{1}{n}} = \frac{n^2+1}{n^3+1} \cdot \frac{n}{1}$

$= \frac{n^3+n}{n^3+1} \xrightarrow{n \to \infty} 1$

$\sum \frac{1}{n}$ Diverges

$\Longrightarrow \sum a_n$ Diverges.

17. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-1)}$

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2(n+1)-1)}{2 \cdot 5 \cdot 8 \cdots (3n-1)(3(n+1)-1)}}{\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 5 \cdot 8 \cdots (3n-1)}} = \frac{2n+2-1}{3n+3-1} = \frac{2n+1}{3n+2} \xrightarrow{n \to \infty} \frac{2}{3}$

$\Rightarrow$ Conv. Abs.

18. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$

#18 $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$ $\qquad \sum \frac{1}{\sqrt{n}}$

Conv. Cond.

Alt. Series

19. $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$

$\frac{\ln n}{\sqrt{n}} > \frac{1}{\sqrt{n}}$ & $\sum \frac{1}{\sqrt{n}}$ fails p-test So not Abs Convergent.

But $\frac{\ln n}{\sqrt{n}} \xrightarrow[L'H]{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2}n^{-\frac{1}{2}}} = \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = \frac{2\sqrt{n}}{n} = \frac{2}{\sqrt{n}} \xrightarrow{n\to\infty} 0$ ✓

Leaving out $a_{n+1} \le a_n$ $\forall n \ge$ some N.

Do a derivative

Do graph of $f(x) = \frac{\ln x}{\sqrt{x}}$

20. $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k} - 1}{k(\sqrt{k} + 1)}$

Intuit $\frac{k^{\frac{1}{3}}}{k^{3/2}} = \frac{1}{k^{7/6}}$ $\quad \frac{9}{6} - \frac{2}{6} = \frac{7}{6}$

$a_n = \frac{\sqrt[3]{k} - 1}{k^{3/2} + k} < \frac{\sqrt[3]{k}}{k^{3/2}} = \frac{1}{k^{7/6}}$

Direct compare to

21. $\sum_{n=1}^{\infty} (-1)^n \cos(1/n^2)$

Diverges

22. $\sum_{k=1}^{\infty} \frac{1}{2 + \sin k}$

Diverges

23. $\sum_{n=1}^{\infty} \tan(1/n)$

$\tan\left(\frac{1}{n}\right) = \frac{\sin\left(\frac{1}{n}\right)}{\cos\left(\frac{1}{n}\right)}$

$\xrightarrow{n \to \text{Big}} \frac{0}{1}$

$\int_1^{\infty} \tan\left(\frac{1}{n}\right) dn$

$\lim_{t\to\infty} \int_1^t \tan\left(\frac{1}{x}\right) dx$

24. $\sum_{n=1}^{\infty} n \sin(1/n)$

$a_n = \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \xrightarrow{n\to\infty} 1$

Diverges by L'H

33. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$

$\left(\left(\frac{n}{n+1}\right)^{n}\right)^{n}$

$\dfrac{\left(\frac{n+1}{n+2}\right)^{(n+1)^2}}{\left(\frac{n}{n+1}\right)^{n^2}}$

$\sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}}$

$=\left(\frac{n}{n+1}\right)^{n}$

34. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+n\cos^2 n}$