11.6 - Absolute and Conditional Convergence: Ratio and Root Tests

11.6 #s 1 - 10, 19 - 26, 31

$\Sigma\, a_n$

The sum of the absolute values:

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

1 Definition A series $\Sigma\, a_n$ is called **absolutely convergent** if the series of absolute values $\Sigma\, |a_n|$ is convergent.

We already know a fair amount about absolutely convergent series, like *p*-series and geometric series.

We're about to learn quite a bit more!

We know, from 11.5, that the alternating series

$$\Sigma\, a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

because its terms a_n converge monotonically to zero. But the sum of its absolute values *also* converges, since

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

$$= \sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

Converges Absolutely

is a convergent p = 2-series.

Non-Example:

alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$ Diverges

is a divergent p = 1-series. Absolutely convergent series abound, next week,
and I wager that the Ratio Test, in this section, will be your best friend after Thanksgiving,
and we learn it, here in 11.6!

Woo-Hoo!

$\boxed{2}$ **Definition** A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

$\boxed{3}$ **Theorem** If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Well, duh.

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$

n	$a_n = \frac{\cos(n)}{n^2}$	$\sum_{k=1}^{n} \frac{\cos(k)}{k^2}$
1	0.540302306	0.540302
2	-0.10403671	0.436266
3	-0.10999917	0.326266
4	-0.04085273	0.285414
5	0.011346487	0.29676
6	0.026671397	0.323432
7	0.01538576	0.338817
8	-0.00227344	0.336544
9	-0.01124852	0.325295
10	-0.00839072	0.316905
11	3.6576E-05	0.316941
12	0.005860097	0.322801
13	0.005369508	0.328171
14	0.000697639	0.328868
15	-0.00337639	0.325492
16	-0.00374086	0.321751
17	-0.00095212	0.320799
18	0.002038015	0.322837
19	0.002738794	0.325576
20	0.001020205	0.326596
21	-0.00124202	0.325354
22	-0.00206603	0.323288

Not Alternating
But is absolutely convergent.

$$-\frac{1}{n^2} \le \frac{\cos(n)}{n^2} \le \frac{1}{n^2}$$

The Ratio Test

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\Sigma\, a_n$.

Examples: $\sum \frac{5^n}{n!}$

The p = 1-series (Harmonic Series) diverges, but the ratio test is no help in showing divergence.

The series, below. Recall, we had an upside-down version of this on homework, that converged (absolutely, now that we know what absolute convergence is). So don't expect this one to converge. Just trying to throws these n-to-the-ns and n-factorials at you. :o)

1 $\sum_{n=1}^{\infty} \frac{5^n}{n!}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{5^{n+1}}{(n+1)!}}{\frac{5^n}{n!}} = \frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n} = \frac{5}{n+1} \xrightarrow{n\to\infty} 0$

$$\frac{n(n-1)(n-2)\cdots(3)(2)}{\boxed{(n+1)}(n)(n-1)\cdots(3)(2)}$$

2 $\sum_{n=1}^{\infty} \frac{1}{n}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{n+1}}{\frac{1}{n}}$

$$= \frac{n}{n+1} \xrightarrow{n\to\infty} 1 \quad \text{Dunno}$$

$\sum_{n=1}^{\infty} \frac{n^n}{n!}$. $\quad \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{\cancel{(n+1)} \cdot (n+1)^n \cdot \cancel{n!}}{\cancel{(n+1)}\cancel{n!} \cdot n^n}$

$$= \frac{(n+1)^n}{n^n} = \frac{n^n + \text{smaller Stuff}}{n^n} \xrightarrow{n\to\infty} 1$$

Inconclusive

3 **8.** $\sum_{n=1}^{\infty} \frac{n!}{100^n}$

$$\frac{(n+1)!}{100^{n+1}} \cdot \frac{100^n}{n!} = \frac{(n+1)}{100} \xrightarrow{n\to\infty} \infty \quad \text{Diverges.}$$

The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

If $L = 1$ in the Ratio Test, don't try the Root Test because L will again be 1. And if $L = 1$ in the Root Test, don't try the Ratio Test because it will fail too.

$$\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$$

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \frac{2n+3}{3n+2} \xrightarrow{n \to \infty} \frac{2}{3} < 1$$

$\Rightarrow$ Abs. Convergent.

Rearrangements

$$a_1 + a_2 + a_5 + a_3 + a_4 + a_{15} + a_6 + a_7 + a_{20} + \ldots$$

We can rearrange a finite sum and not change its value. This is the commutative law of addition.

Rearranging won't change the value of an absolutely convergent series. But you can change a conditionally convergent series (for instance, alternating harmonic series) to any number you want, by rearranging the positive and negative terms, which is kind of mind-blowing. This is a standard result that you prove in advanced calculus.

But *conditionally* convergent means it's *not* absolutely convergent. So, basically, you take only the positive terms and let them build up above the number you want to hit. As soon as you go over, you add a negative term to drag you down below. Then add positive terms to take you above, and so on.

$$\boxed{6} \quad 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{8} + \cdots = \ln 2$$

If we multiply this series by $\frac{1}{2}$, we get

$$\tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{6} - \tfrac{1}{8} + \cdots = \tfrac{1}{2} \ln 2$$

Inserting zeros between the terms of this series, we have

$$\boxed{7} \quad 0 + \tfrac{1}{2} + 0 - \tfrac{1}{4} + 0 + \tfrac{1}{6} + 0 - \tfrac{1}{8} + \cdots = \tfrac{1}{2} \ln 2$$

Now we add the series in Equations 6 and 7:

$$\boxed{8} \quad 1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \cdots = \tfrac{3}{2} \ln 2$$

1. What can you say about the series $\Sigma\, a_n$ in each of the following cases?

(a) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 8$ Diverges

(b) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0.8$ Converges

(c) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ My don't know!

2–30 Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{(n+1)^2}\cdot\frac{n^2}{2^n} = 2\cdot\frac{n^2}{(n+1)^2} \xrightarrow{n\to\infty} 2$$

Diverges

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{5^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{5^{n+1}}\cdot\frac{5^n}{n} = \frac{n+1}{5n} \xrightarrow{n\to\infty} \frac{1}{5} < 1 \quad \text{Converges}$$

4. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{n}{n^2+4}$

$$\frac{n+1}{(n+1)^2+4}\cdot\frac{n^2+4}{n} = \frac{n^3+\text{smaller}}{n^3+\text{smaller}} \xrightarrow{n\to\infty} 1 \quad ?$$

But limit comparison with $b_n = \frac{1}{n}$

$$\left|\frac{a_n}{b_n}\right| = \frac{\frac{n}{n^2+4}}{\frac{1}{n}} = \frac{n}{n^2+4}\cdot\frac{n}{1} = \frac{n^2}{n^2+4} \xrightarrow{n\to\infty} 1$$

$\sum |a_n|$ Diverges, since $\sum b_n$ diverges & $\left|\frac{a_n}{b_n}\right| \xrightarrow{n\to\infty} 1$

Converges Conditionally by alternating series test.

5. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{5n+1}$

$b_n = \frac{1}{n}$

$$\left|\frac{a_n}{b_n}\right| = \frac{1}{5n+1}\cdot\frac{n}{1} \xrightarrow{n\to\infty} \frac{1}{5}$$

so they both converge/diverge the same.

Conditional (Alt. Series)

6. $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{3^n}$$

$$= \frac{3 \cdot (2n+1)!}{(2n+2+1)!} = \frac{3 \cdot (2n+1)!}{(2n+3)!} = \frac{3 \cdot \cancel{(2n+1)!}}{(2n+3)(2n+2)\cancel{(2n+1)!}}$$

$$= \frac{3}{(2n+3)(2n+2)} \xrightarrow{n\to\infty} 0 \longrightarrow \boxed{\text{Abs. Conv.}}$$

7. $\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k$

$$\frac{(k+1)\left(\frac{2}{3}\right)^{k+1}}{k\left(\frac{2}{3}\right)^k} = \frac{(k+1)\left(\frac{2}{3}\right)}{k} \xrightarrow{k\to\infty} \boxed{\tfrac{2}{3} \quad \text{conv. abs.}}$$

8. $\sum_{n=1}^{\infty} \frac{n!}{100^n}$ See theory talk (Diverges)

$$\frac{1000000}{1000001}$$

9. $\sum_{n=1}^{\infty} (-1)^n \frac{(1.1)^n}{n^4}$

$$\frac{(1.1)^{n+1}}{(n+1)^4} \cdot \frac{n^4}{(1.1)^n} = \frac{1.1\, n^4}{(n+1)^4} \xrightarrow{n\to\infty} 1.1$$

Alt. Series Test.

$$\frac{(1.1)^n}{n^4} \xrightarrow{n\to\infty} \infty$$

Exponential

$$\frac{a^x}{x^{55}} \xrightarrow{x\to\infty} \infty \text{ if } a>1$$

$$\frac{a^n}{n^{\text{constant}}} \xrightarrow{n\to\infty} \infty \text{ if } a>1$$

polynomial (Power Function)

~~To~~ L'Hôpital's if you need convincing

10. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$

$\boxed{\text{Intuit: } \sum \frac{1}{n^{1/2}} \quad \text{Limit Comparison}}$

$\sum (-1)^n \frac{1}{n^{1/2}}$ converges conditionally, Alt. Series Test

$\frac{n}{\sqrt{n^3+2}} > 0 \quad \forall\, n \geq 1$

$\xrightarrow{n\to\infty} 0$ ✓

$\frac{x}{\sqrt{x^3+2}} = \frac{\cancel{x}}{\cancel{x}\sqrt{x+\frac{2}{x^2}}} = \left(x+\frac{2}{x^2}\right)^{-\frac{1}{2}} = y \Rightarrow$

$y' = -\frac{1}{2}\left(x+\frac{2}{x^2}\right)^{-\frac{3}{2}}\left(1-4x^{-3}\right) < 0$ for big x.

$\boxed{\text{Converges Conditionally}}$

19. $\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$

limit comparison to

$\sum \frac{1}{n!}$

$\left|\frac{\cos\left(\frac{n\pi}{3}\right)}{n!}\right| \leq \frac{1}{n!}$ &

$\sum \frac{1}{n!}$ converges

20. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$

$\left|\frac{(-2)^n}{n^n}\right| = \frac{2^n}{n^n} = \left(\frac{2}{n}\right)^n$

$\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{2}{n}\right)^n} = \frac{2}{n} \xrightarrow{n\to\infty} 0$

$\Rightarrow$ Abs. Conv.

21. $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$

$\sqrt[n]{|a_n|} = \frac{n^2+1}{2n^2+1} \xrightarrow{n\to\infty} \frac{1}{2} < 1$

$\Rightarrow$ Abs. Conv.

22. $\displaystyle\sum_{n=2}^{\infty}\left(\frac{-2n}{n+1}\right)^{5n}$

Diverges

$\left|\frac{-2n}{n+1}\right| \xrightarrow{n\to\infty} 2$

So $\left(\frac{2n}{n+1}\right)^{5n} \xrightarrow{n\to\infty} \infty$!

23. $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$

$a_n = \left(1+\frac{1}{n}\right)^{n\cdot n} = \left(\left(1+\frac{1}{n}\right)^n\right)^n \xrightarrow{n\to\infty} e^n$

Divergent

$\left(1+\frac{1}{n}\right)^n \xrightarrow{n\to\infty} e$

$2(n+1) = 2n+2$, silly!

24. $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2n+2)!}{((n+1)!)^2}\cdot\frac{(n!)^2}{(2n)!} = \frac{(2n+2)\ n!\ n!}{(n+1)!\ (n+1)!} = \frac{2n+2}{(n+1)(n+1)} \xrightarrow{n\to\infty} 0$

Abs. Conv. !

25. $\displaystyle\sum_{n=1}^{\infty}\frac{n^{100}100^n}{n!}$

$\sqrt[n]{n!} = ?$

$\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{100}\,100^{n+1}}{(n+1)!}\cdot\frac{n!}{n^{100}\,100^n} = \frac{(n+1)^{100}}{n^{100}}\cdot\frac{100^{n+1}}{100^n}\cdot\frac{n!}{(n+1)!}$

$= \left(\frac{n+1}{n}\right)^{100}\cdot 100\cdot\frac{1}{n+1}$

$\xrightarrow{n\to\infty} 1\cdot 100\cdot 0 = 0$

Converges Absolutely

26. $\displaystyle\sum_{n=1}^{\infty}\frac{2^{n^2}}{n!}$

Diverges

$2^{n^2} = 2^{n\cdot n} = (2^n)^n > n^n$, eventually

and $\frac{n^n}{n!} \xrightarrow{n\to\infty} \infty$

Diverges

$\frac{2^{(n+1)^2}}{(n+1)!}\cdot\frac{n!}{2^{n^2}}$

$= \frac{2^{n^2+2n+1}}{(n+1)\ 2^{n^2}} = \frac{2^{n^2}\cdot 2^{2n}\cdot 2^1}{(n+1)\left(2^{n^2}\right)} = \frac{2^{2n+1}}{n+1} \xrightarrow{n\to\infty} \infty$

31. The terms of a series are defined recursively by the equations

$$a_1 = 2 \qquad a_{n+1} = \frac{5n + 1}{4n + 3}\, a_n$$

Determine whether $\Sigma\, a_n$ converges or diverges.