Section 11.4 COMPARISON TESTS
Section 11.4 #s 1 - 20, 29 - 37

The Comparison Test Suppose that Σa_n and Σb_n are series with positive terms.
(i) If Σb_n is convergent and $a_n \leq b_n$ for all n, then Σa_n is also convergent.
(ii) If Σb_n is divergent and $a_n \geq b_n$ for all n, then Σa_n is also divergent.

Does $\sum \frac{1-\frac{1}{n}}{n^2+11} = \sum a_n$ converge?

Define $b_n = \frac{1}{n^2}$, then $a_n = \frac{1-\frac{1}{n}}{n^2+11} \leq \frac{1}{n^2} = b_n$ & $\sum \frac{1}{n^2}$ converges $(p=2)$

$\Rightarrow \sum a_n$ converge

Does $\sum \frac{1+\frac{1}{n}}{n-11} = \sum a_n$

Define $b_n = \frac{1}{n} \leq \frac{1+\frac{1}{n}}{n-11}$ $(n > 11)$ $= a_n$

& $\sum b_n$ diverges $(p=1) \Rightarrow \sum a_n$ diverges.

Trouble with direct comparison:

$\sum \frac{1+\frac{1}{n}}{n^2-11}$ Intuition says it's like $\sum \frac{1}{n^2}$, but

$\frac{1+\frac{1}{n}}{n^2-11}$ isn't easily compared to $\frac{1}{n^2}$, because

$a_n = \frac{1+\frac{1}{n}}{n^2-11} > \frac{1}{n^2} = b_n$ = logical (intuitive) comparison.

Tough to make direct comparison

Tricks $\frac{a-b}{c+d} < \frac{a}{c}$ for convergence

$\frac{a+b}{c-d} > \frac{a}{c}$ for divergence

The Limit Comparison Test Suppose that Σa_n and Σb_n are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

$a_n = \frac{1+\frac{1}{n}}{n^2-11}$, $b_n = \frac{1}{n^2}$ $\sum \frac{1+\frac{1}{n}}{n^2-11}$ Intuit $\sum \frac{1}{n^2}$ which converges.

Then $\frac{a_n}{b_n} = \frac{\frac{1+\frac{1}{n}}{n^2-11}}{\frac{1}{n^2}} = \left(\frac{1+\frac{1}{n}}{n^2-11}\right)\left(\frac{n^2}{1}\right) = \frac{n^2+n}{n^2-11} \xrightarrow{n\to\infty} 1 = C$

Alternate method

$\frac{n^2+n}{n^2-11} = \frac{n^2(1+\frac{1}{n})}{n^2(1-\frac{11}{n^2})} = \frac{1+\frac{1}{n}}{1-\frac{11}{n^2}} \xrightarrow{n\to\infty} \frac{1}{1} = 1 = C$

$\Rightarrow \sum a_n$ converges, b/c $\sum b_n$ converges.

$\rightarrow 0$

The Limit Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

40. (a) Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms and $\Sigma\, b_n$ is convergent. Prove that if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$$

then $\Sigma\, a_n$ is also convergent.

This is when the limit turns out even better!

41. (a) Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms and $\Sigma\, b_n$ is divergent. Prove that if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$$

then $\Sigma\, a_n$ is also divergent.

Even MOAR obvious it diverges!

Estimating Sums

Estimate $\sum_{n=1}^{\infty} \frac{1-\frac{1}{n}}{n^2+1}$ by summing the 1^{st} 10 terms, BORING

Give an upper bound on the error.

Pretty cool.

$$R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots \leq \int_n^{\infty} \frac{1-\frac{1}{x}}{x^2+1}\,dx \leq \int_n^{\infty} \frac{1}{x^2}\,dx$$

$$T_n = t - t_n = b_{n+1} + b_{n+2} + \cdots \leq \int_n^{\infty} \frac{dx}{x^2}$$

$$b_n = \frac{1}{n^2}$$

$$S_{10} = \sum_{n=1}^{10} \frac{1-\frac{1}{n}}{n^2+1} \approx 0.1781194879 \pm \frac{1}{10}$$

and $\int_{10}^{\infty} \frac{dx}{x^2} = \frac{1}{10}$ = upper bound on error

1. Suppose $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms and $\Sigma\, b_n$ is known to be convergent.
(a) If $a_n > b_n$ for all n, what can you say about $\Sigma\, a_n$? Why?
(b) If $a_n < b_n$ for all n, what can you say about $\Sigma\, a_n$? Why?

(a) Nothing. $\Sigma\, a_n$ may con- or diverge
Dunno.
(b) $\Sigma\, a_n$ converges, Comparison Test

2. Suppose $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms and $\Sigma\, b_n$ is known to be divergent.
(a) If $a_n > b_n$ for all n, what can you say about $\Sigma\, a_n$? Why?
(b) If $a_n < b_n$ for all n, what can you say about $\Sigma\, a_n$? Why?

(a) $\Sigma\, a_n$ diverges
comparison Test.

(b) Dunno.

3–32 Determine whether the series converges or diverges.

3. $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ **4.** $\sum_{n=2}^{\infty} \frac{n^3}{n^4-1}$

(3) $a_n = \frac{n}{2n^3+1} < b_n = \frac{n}{2n^3} = \frac{1}{2n^2}$

& $\sum \frac{1}{2n^2} = \frac{1}{2}\sum \frac{1}{n^2}$ converges

so $\boxed{\sum a_n \text{ converges.}}$

(4) is "like" $\sum \frac{n^3}{n^4} = \sum \frac{1}{n}$ diverges

$a_n = \frac{n^3}{n^4-1} > \frac{n^3}{n^4} = \frac{1}{n}$ &

$\sum \frac{1}{n}$ diverges

$\Rightarrow \boxed{\sum \frac{n^3}{n^4-1} \text{ also diverges}}$

5. $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$

$a_n = \frac{n+1}{n^{3/2}} > \frac{n}{n^{3/2}} = \frac{1}{n^{\frac{1}{2}}} = b_n$

& $\sum b_n$ diverges $\Rightarrow$

$\boxed{\sum a_n \text{ diverges}}$

(5b) $\sum \frac{n-1}{n\sqrt{n}}$ diverges, but hard to make it bigger than anything obvious.

$\frac{n}{n\sqrt{n}} > \frac{n-1}{n\sqrt{n}}$ so using it to show divergence, directly is tough.

This would require LIMIT comparison to $\sum \frac{1}{n^{1/2}}$

6. $\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$ $\frac{n}{n^2\sqrt{n}} = \frac{n}{n^{5/2}} = \frac{1}{n^{3/2}} = b_n$

$a_n = \frac{n-1}{n^{5/2}} < \frac{n}{n^{5/2}} = \frac{1}{n^{3/2}} = b_n$ &

$\sum b_n$ converges $\Rightarrow$

$\boxed{\sum a_n \text{ converges}}$

7. $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$

$\sum \frac{9^n}{10^n-3}$ LIMIT comparison

$a_n = \frac{9^n}{3+10^n} < \frac{9^n}{10^n} = \left(\frac{9}{10}\right)^n = b_n$

$\sum b_n$ converges, b/c geometric series,

$r = \frac{9}{10}$ & $|r| < 1 \Rightarrow$ converges

$\boxed{\sum a_n \text{ converges}}$

8. $\sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$ compare to divergent

$\sum \left(\frac{6}{5}\right)^n$

$\sum \frac{5^n}{6^n+3}$ would require Limit comparison to show convergence

$a_n = \frac{6^n}{5^n-1} > \frac{6^n}{5^n} = \left(\frac{6}{5}\right)^n$

$\boxed{\sum a_n \text{ Diverges}}$

$\sum \frac{6^n}{5^n+1}$ would require LIMIT Comparison to show divergence

9. $\sum_{k=1}^{\infty} \frac{\ln k}{k}$

Eventually, $\ln k > 1$

Take $k \geq 3 \Rightarrow$

$\ln k > 1$

$b_n = \frac{1}{k}$

& $\sum a_n$ will diverge

10. $\sum_{k=1}^{\infty} \frac{k \sin^2 k}{1 + k^3}$

$\sin^2 k \leq 1$

$a_n = \frac{k \sin^2 k}{k^3+1} \leq \frac{k}{k^3+1} < \frac{k}{k^3} = \frac{1}{k^2} = b_n$

$\sum b_n$ converges $\Rightarrow$

$\sum a_n$ converges

11. $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}}{\sqrt{k^3 + 4k + 3}}$

$a_n = \frac{k^{1/3}}{(k^3+4k+3)^{\frac{1}{2}}} \leq \frac{k^{\frac{1}{3}}}{(k^3)^{1/2}} = \frac{k^{\frac{1}{3}}}{k^{3/2}} = \frac{1}{k^{7/6}} = b_n$

$\frac{3}{2} - \frac{1}{3} = \frac{9-2}{6} = \frac{7}{6}$

& $\sum b_n$ converges $\Rightarrow$

$\sum a_n$ converges.

12. $\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$

$\frac{(2k)(k^2)}{(k)(k^2)^2} = \frac{2k^3}{k^5} = \frac{2}{k^2} = b_n$

$a_n \leq \frac{2}{k^2} = b_n$ &

$\sum b_n$ converges $\Rightarrow$

$\sum a_n$ converges.

13. $\sum_{n=1}^{\infty} \frac{\arctan n}{n^{1.2}}$

$a_n = \frac{\arctan(n)}{n^{1.2}} < \frac{\frac{\pi}{2}}{n^{1.2}} = b_n$

$\sum b_n = \sum \frac{\frac{\pi}{2}}{n^{1.2}} = \frac{\pi}{2} \sum \frac{1}{n^{1.2}}$

converges

$\Rightarrow \sum a_n$ converges

14. $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$

$b^n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$

$\sum b_n$ diverges

$a_n = \frac{\sqrt{n}}{n-1} > \frac{\sqrt{n}}{n} = b_n$

$\Rightarrow \sum a_n$ diverges

15. $\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n - 2}$

$b_n = \frac{4^{n+1}}{3^n} = \frac{4 \cdot 4^n}{3^n} = 4\left(\frac{4}{3}\right)^n$

$r = \frac{4}{3} > 1 \Rightarrow \sum b_n$ diverges

Geometric $r > 1$

$a_n = \frac{4^{n+1}}{3^n - 2} > \frac{4^{n+1}}{3^n} = b_n$

$\boxed{\sum a_n \text{ diverges}}$

16. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4 + 1}}$

$b_n = \frac{1}{\sqrt[3]{3n^4}} = \frac{1}{\sqrt[3]{3}\sqrt[3]{n^4}}$

$= \frac{1}{\sqrt[3]{3}} \cdot \frac{1}{n^{4/3}}$ &

p-test

$\frac{1}{\sqrt[3]{3}} \sum \frac{1}{n^{4/3}}$ converges.

want to compare

$a_n = \frac{1}{\sqrt[3]{3n^4+1}} < \frac{1}{\sqrt[3]{3n^4}}$

$\Rightarrow \boxed{\sum a_n \text{ converges}}$

17. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ Harmonic

$b_n = \frac{1}{\sqrt{n^2}} = \frac{1}{n}$ & $\sum b_n$ diverges
p = 1 - test

Trouble is,
a_n gets SMALLER when I throw away "+1" in denominator.

$a_n < b_n$, BUT

$$\frac{a_n}{b_n} = \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{\sqrt{n^2}}} = \frac{1}{\sqrt{n^2+1}} \cdot \frac{\sqrt{n^2}}{1}$$

$$= \frac{n}{\sqrt{n^2(1+\frac{1}{n^2})}} = \frac{n}{\sqrt{n^2}\sqrt{1+\frac{1}{n^2}}}$$

$$= \frac{n}{|n|\sqrt{1+\frac{1}{n^2}}} = \frac{n}{n\sqrt{1+\frac{1}{n^2}}} = \frac{1}{\sqrt{1+\frac{1}{n^2}}} \xrightarrow{n\to\infty} 1$$

$\Rightarrow \sum a_n$ diverges

19. $\sum_{n=1}^{\infty} \frac{1+4^n}{1+3^n}$

$a_n = \frac{1}{1+3^n} + \frac{4^n}{1+3^n}$

$b_n = \frac{4^n}{3^n} = \left(\frac{4}{3}\right)^n$ & $\sum b_n$ diverges.

$\frac{1+4^n}{1+3^n} > \frac{4^n}{3^n+1} > ???$

But $\frac{4^n}{3^n+1} < \frac{4^n}{3^n+1}$ & we want $a_n > b_n$ to show divergence!

So $\frac{a_n}{b_n} = \frac{\frac{4^n+1}{3^n+1}}{\frac{4^n}{3^n}}$

$$= \frac{4^n+1}{3^n+1} \cdot \frac{3^n}{4^n}$$

$$= \frac{4^n(1+4^{-n})}{3^n(1+3^{-n})} \cdot \frac{3^n}{4^n}$$

$$= \frac{1+4^{-n}}{1+3^{-n}} \xrightarrow{n\to\infty} 1$$

$\Rightarrow$ $\boxed{\sum a_n \text{ diverges}}$

b/c $\sum b_n$ is divergent (Geometric, $r = \frac{4}{3} > 1$)

20. $\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$

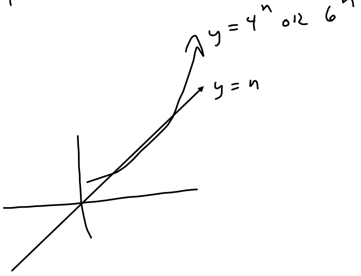
$b_n = \left(\frac{4}{6}\right)^n = \left(\frac{2}{3}\right)^n$

$$\frac{a_n}{b_n} = \frac{\frac{n+4^n}{n+6^n}}{\frac{4^n}{6^n}} = \frac{\frac{4^n+n}{6^n+n}}{\frac{4^n}{6^n}}$$

$$= \frac{4^n+n}{6^n+n} \cdot \frac{6^n}{4^n} = \frac{4^n(1+\frac{n}{4^n})}{6^n(1+\frac{n}{6^n})} \cdot \frac{6^n}{4^n} =$$

$$= \frac{1+\frac{n}{4^n}}{1+\frac{n}{6^n}} \xrightarrow{n\to\infty} 1$$

$\frac{n}{4^n} \to 0$, $\frac{n}{6^n} \to 0$



$\Rightarrow \sum a_n$ converges, since $\sum b_n$ is Geometric

$0 < r = \frac{4}{6} = \frac{2}{3} < 1$

29. $\sum_{n=1}^{\infty} \frac{1}{n!}$

$n! = \underbrace{n(n-1)(n-2)\cdots(3)(2)(1)}_{n-1 \text{ factors}}$

$\geq \underbrace{2\cdot 2\cdot 2\cdot \cdots 2\cdot 2}_{n-1} = 2^{n-1}$

$b^n = \frac{1}{2^{n-1}} > \frac{1}{n!} = a_n$

& $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}$ is convergent. $|r|<1$

Geometric, $r = \frac{1}{2} < 1$

& r is positive $0 < r = \frac{1}{2} <$

30. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$\frac{n(n-1)(n-2)\cdots(3)(2)(1)}{n\cdot n\cdot n \cdots n\cdot n\cdot n}$

$\leq \frac{1\cdot 1\cdot 1 \cdots 2\cdot 1}{n\cdot n} = \frac{2}{n^2}$

& $\sum b_n = \sum \frac{2}{n^2}$ converges, so

$\sum a_n = \sum \frac{n!}{n^n}$ converges.

31. $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ diverges, because $\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = 1$

and $\sum b_n = \sum \frac{1}{n}$ diverges.

(Harmonic)

$n=1=p$-test

Relabel : $\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = 1$

Recall: $\lim_{x\to 0} \frac{\sin x}{x} = 1$

32. $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

Intuition: $\frac{1}{n} = b_n$

$\frac{\frac{1}{n^{1+1/n}}}{\frac{1}{n}} = \frac{1}{n^{1+1/n}}\cdot\frac{n}{1}$

$= \frac{1}{\cancel{n}\cdot n^{1/n}}\cdot\frac{\cancel{n}}{1} = \frac{1}{n^{1/n}} \xrightarrow{n\to\infty}$

$\frac{1}{1} = 1$ So both diverge.

$n^{\frac{1}{n}} \xrightarrow{n\to\infty} 1$, by L'Hopital's

(I use $\log(y)$ for this)

$y = n^{\frac{1}{n}} \Rightarrow$

$\ln(y) = \ln(n^{1/n}) = \frac{1}{n}\ln(n)$

$= \frac{\ln(n)}{n} \xrightarrow[L'H]{n\to\infty}$

$\frac{\frac{1}{n}}{1} = \frac{1}{n} \xrightarrow{n\to\infty} 0$

$\lim_{n\to\infty}(\ln(y)) = 0$

$\ln(\lim_{n\to\infty}(y)) = 0$

$\lim_{n\to\infty}(y) = 1$

#32 Diverges since $\sum \frac{1}{n}$ diverges.

33–36 Use the sum of the first 10 terms to approximate the sum of the series. Estimate the error. $n=10$

$$\sum_{k=1}^{10} \frac{1}{\sqrt{k^4+1}} \approx 1.248558569$$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4+1}} = S_n + R_n = \sum_{k=1}^{10} \frac{1}{\sqrt{k^4+1}} + \sum_{k=n+1}^{\infty} \frac{1}{\sqrt{k^4+1}}$

$$\lim_{t\to\infty}\left(\int_{10}^{t} \frac{1}{x^2}\,dx\right) = \boxed{\frac{1}{10} \geq R_n}$$

$R_n = a_{n+1} + a_{n+2} + \cdots$

$a_n = \dfrac{1}{\sqrt{n^4+1}}$

$b_n = \dfrac{1}{\sqrt{n^4}} = \dfrac{1}{n^2} > \dfrac{1}{\sqrt{n^4+1}}$

$t_n = b_{n+1} + b_{n+2} + \cdots \leq \displaystyle\int_n^{\infty} \frac{dx}{x^2}$

$\sum b_n = T_n + t_n \leq T_n + \displaystyle\int_n^{\infty} \frac{dx}{x^2}$

$\&\ t_n \geq R_n$

is our ceiling on t_n which is our ceiling R_n.

34. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}$ $\qquad R_n \leq t_n \leq \displaystyle\int_{10}^{\infty} \frac{1}{x^3}\,dx = .005$

$$\sum_{k=1}^{10} \frac{\sin^2(k)}{k^3} \approx 0.8325298015$$

$$\sin(1)^2 + \frac{1}{8}\sin(2)^2 + \frac{1}{27}\sin(3)^2 + \frac{1}{64}\sin(4)^2 + \frac{1}{125}\sin(5)^2 + \frac{1}{216}\sin(6)^2 + \frac{1}{343}\sin(7)^2$$
$$+ \frac{1}{512}\sin(8)^2 + \frac{1}{729}\sin(9)^2 + \frac{1}{1000}\sin(10)^2$$

35. $\sum_{n=1}^{\infty} 5^{-n}\cos^2 n$

$a_n = 5^{-n}\cos^2 n \le 5^{-n} = b_n$

$R_{10} \le t_{10} \le \int_{10}^{\infty} 5^{-x}\,dx \approx 6.362469731 \times 10^{-8}$

$\sum_{k=1}^{10} 5^{-k}\cos^2(k) \approx$ 0.07392930340

36. $\sum_{n=1}^{\infty} \frac{1}{3^n + 4^n}$

$a_n = \frac{1}{3^n+4^n} \le \frac{1}{3^n}$ or $\frac{1}{4^n}$, but tighter estimate would be $\frac{1}{3^n+3^n} = \frac{1}{2\cdot 3^n} \ge \frac{1}{3^n+4^n}$

$R_{10} \le t_n \le \frac{1}{2}\int_{10}^{\infty} 3^{-x}\,dx \approx$ 0.000007707490613

$\sum_{k=1}^{10} \frac{1}{3^k+4^k} = \frac{13909205036362862190076475 2}{70290328957429914091892087 5} \approx 0.1978822015$

37. The meaning of the decimal representation of a number $0.d_1d_2d_3\ldots$ (where the digit d_i is one of the numbers 0, 1, 2, ..., 9) is that

$$0.d_1d_2d_3d_4\ldots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{d_4}{10^4} + \cdots = d_1 \cdot \frac{1}{10} + d_2 \cdot \left(\frac{1}{10}\right)^2 + d_3 \left(\frac{1}{10}\right)^3 + d_4 \left(\frac{1}{10}\right)^4 + \cdots$$

Show that this series always converges.

Observe $d_k \leq 9 \ \forall k$

So, $\frac{d_k}{10^k} = a_k \leq b_k = \frac{9}{10^k}$

& $\sum b_k = \sum \frac{9}{10^k} = 9 \sum \left(\frac{1}{10}\right)^k$ is convergent, geometric $r = \frac{1}{10}$