11.3 The Integral Test and Estimates of Sums

S 11.3 #s 1 - 6, 9, 10, 12, 13, 15, 16 - 19, 22, 27 - 29, 34, 35 38,

The Integral Test and Estimates of Sums

We begin by investigating the series whose terms are the reciprocals of the squares of the positive integers:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

There's no simple formula for the sum s_n of the first n terms, but the computer-generated table of values given to the right suggests that the partial sums are approaching a number near 1.64 as $n \to \infty$ and so it looks as if the series is convergent.

n^{th} partial sums.

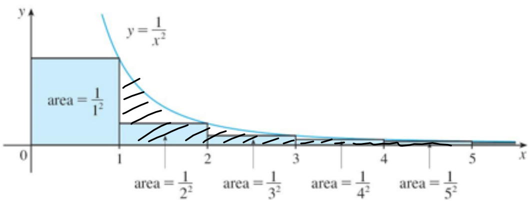
n	$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$\{S_n\} = \left\{ \sum_{k=1}^{n} \frac{1}{k^2} \right\}$ is a sequence of partial sums that is bounded, above, and montone increasing, since we're adding a positive term to the previous partial sum to obtain the next term in the sequence of partial sums, i.e., Bdd

$$S_{n+1} = \sum_{k=1}^{n+1} \frac{1}{k^2} = \left(\sum_{k=1}^{n} \frac{1}{k^2} \right) + \frac{1}{(n+1)^2} > \sum_{k=1}^{n} \frac{1}{k^2} = S_n .$$

So, all this cleverness, and we still don't (and won't) ever know exactly what the sum of the infinite series is! We're *finally* doing some real analysis (and some Real Analysis). We've established only that the sum exists!



$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \sum_{n=2}^{\infty} \frac{1}{n^2} < \frac{1}{1^2} + \int_{x=1}^{\infty} \frac{1}{x^2}\,dx = 1 + 1 = 2$$

The bit of awkwardness with the $\frac{1}{1^2}$ term is needed, because

$\int_{x=0}^{1} \frac{1}{x^2}\,dx$ is unbounded

and thus of no help in our putting a ceiling on the series. So we handle the first term 'by hand.' The main thing is how the series fits snugly *under* the $\frac{1}{x^2}$ curve the rest of the way.

The *hard* part, *always*, is getting a handle on the infinite number of terms in the *tail* of the series.

Summary: We proved the series converged by comparing it to an integral that is...
1. BIGGER than the series, and
2. CONVERGES.

In this book, we have convergence by the Monotonic Sequence Theorem.

Bdd above & increasing ⇒ Converges
Bdd below & decreasing ⇒ Converges

We show that

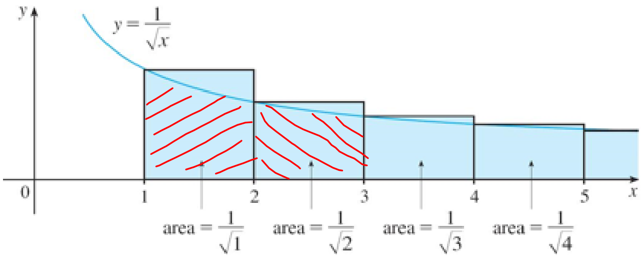
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots$$

is *bigger* than an integral that *diverges*. This will tell us that the series diverges!

Quick numerical investigation suggests divergence:

n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

And graphically, the sum is bigger than the area under $\frac{1}{\sqrt{x}}$



So, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_{x=1}^{\infty} \frac{1}{\sqrt{x}}\,dx$, in a manner of speaking.

$$\int_1^4 \frac{1}{\sqrt{x}}\,dx < \sum_{k=1}^{4} \frac{1}{\sqrt{k}} = S_4$$

$$S_4 \in \{S_n\}$$

More properly speaking, define *another* sequence,

$B_n = \int_{x=1}^{n} \frac{1}{\sqrt{x}}\,dx$. Then $S_n > B_n$ and the sequence of integrals

$\{B_n\}$ is clearly divergent, since we know, from improper integrals,

that $\lim_{t\to\infty} \int_{x=1}^{t} \frac{1}{\sqrt{x}}\,dx = \infty$

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(a) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

$a_n = \frac{1}{\sqrt{n}} = f(n)$

$f(x)$

$\sum a_n$ converges iff $\Longleftrightarrow$ $\int_1^{\infty} f(x)\,dx$ converges

And, Finally....

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called the ***p*-series**.

$\boxed{1}$ The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

So, what's the big deal about these series? Well, they get very close to their final sum, very quickly. The n-tail of the series - also called the *Remainder* - is defined to be the difference between the infinite sum and its n^{th} partial sum.

$$R_n = S - S_n = a_{n+1} + a_{n+2} + a_{n+3} + \ldots.$$

$$S = \underbrace{(a_1 + a_2 + \cdots + a_{n-1} + a_n)}_{S_n} + \underbrace{(a_{n+1} + \cdots)}_{R_n}$$

the n-tail

Our job - should we choose to accept it - is to get estimates on the remainder, and inform us as to how close we are to the infinite series, if we chop it off at some finite value n.

Error for S_n $= R_n$

we use $\int_{n+1}^{\infty}$ & $\int_{n}^{\infty}$ to estimate R_n.

$S \approx S_n$ is the idea, here, and 'puters can spit out those partial sums in nothin' flat!

And the improper integral gives us a floor and a ceiling for the error!

See Taylor & Maclaurin Series

$$R_n = a_{n+1} + a_{n+2} + \cdots \leqslant \int_n^{\infty} f(x)\,dx$$

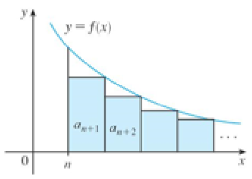


Figure 3

$$R_n = a_{n+1} + a_{n+2} + \cdots \geqslant \int_{n+1}^{\infty} f(x)\,dx$$

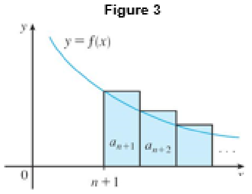


Figure 4

2 Remainder Estimate for the Integral Test Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geqslant n$ and $\Sigma\, a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leqslant R_n \leqslant \int_n^{\infty} f(x)\,dx$$

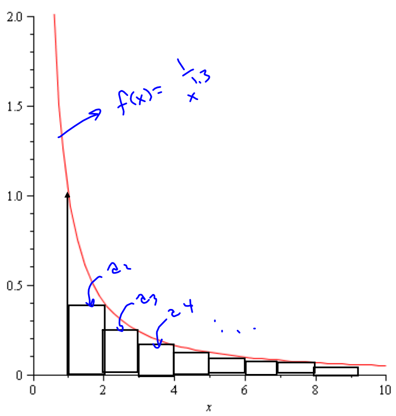
$$S_n + R_n = \quad S \quad = S_n + R_n$$

$$\boxed{3} \quad s_n + \int_{n+1}^{\infty} f(x)\,dx \leqslant s \leqslant s_n + \int_n^{\infty} f(x)\,dx$$

1. Draw a picture to show that

$$\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^{\infty} \frac{1}{x^{1.3}}\, dx$$

What can you conclude about the series?

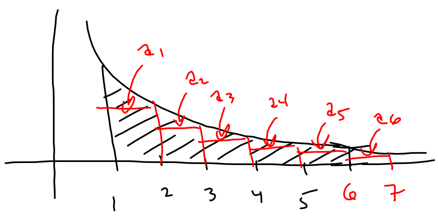


Since $\int_1^{\infty} \frac{dx}{x^{1.3}}$ converges,

So does $\sum_{n=1}^{\infty} \frac{1}{n^{1.3}}$

2. Suppose f is a continuous positive decreasing function for $x \geqslant 1$ and $a_n = f(n)$. By drawing a picture, rank the following three quantities in increasing order:

$$\int_1^6 f(x)\,dx \;>\; \sum_{i=1}^{5} a_i \;>\; \sum_{i=2}^{6} a_i$$



3–8 Use the Integral Test to determine whether the series is convergent or divergent.

3. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$ Diverges $p = \frac{1}{5} < 1$

4. $\sum_{n=1}^{\infty} \frac{1}{n^5}$ converges $p = 5 > 1$

5. $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^3}$ $\sum \frac{1}{(2n)^3}$, basically

Limit Comparison test clobbers #5 & 6

6. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$ $\sum \frac{1}{\sqrt{n}}$, basically

#6 $u = n+4$ $u = x+4$

$\int_1^{\infty} \frac{dx}{\sqrt{x+4}}$

5 w/o 11.4 concepts

$\int_1^{\infty} \frac{1}{(2x+3)^3} dx$:

$\int_1^{t} \frac{1}{(2x+3)^3} dx = \frac{1}{2}\int_1^{t} (2x+3)^{-3} \cdot 2dx$

$u = 2x+3$
$du = 2dx$

$= \frac{1}{2}\left[-\frac{1}{2}(2x+3)^{-2}\right]_1^t$

$= -\frac{1}{4}\left[(2t+3)^{-2} - (2(1)+3)^{-2}\right]$

$= -\frac{1}{4}\left[(2t+3)^{-2} - \frac{1}{25}\right]$

$\xrightarrow{t \to \infty} -\frac{1}{4}\left[-\frac{1}{25}\right] = \frac{1}{100}$

Converges!

9–26 Determine whether the series is convergent or divergent.

9. $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$

Converges

$p = \sqrt{2} > 1$

10. $\sum_{n=3}^{\infty} n^{-0.9999} = \sum \frac{1}{n^{.9999}}$

Diverges

$p = .9999 < 1$

12. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots$

$2\sqrt{2} = 2(2)^{\frac{1}{2}} = 2^1 (2)^{\frac{1}{2}}$

$= 2^{1+\frac{1}{2}} = 2^{\frac{3}{2}} \Rightarrow$

$\boxed{p = \frac{3}{2} > 1 \quad \text{converges}}$

$3\sqrt{3}$

$3 \cdot 3^{\frac{1}{2}} = 3^{\frac{3}{2}}$

13. $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots + \frac{1}{2n-1} + \cdots$

1 2 3 4

$2n+1$

$2n-1$

$1 = \frac{1}{1} = \frac{1}{2(1)-1}$ ✓

$\int_1^\infty \frac{dx}{2x-1} \; : \; \frac{1}{2}\int_1^t \frac{2\,dx}{2x-1} = \frac{1}{2}\int_*^* u^{-1}\,du$

$= \frac{1}{2}\ln|u|\Big]_*^* = \frac{1}{2}\ln|2x-1|\Big]_1^t$

$\frac{1}{2}\ln|2t-1| - \frac{1}{2}\ln|1|$

$= \frac{1}{2}\ln|2t-1| \xrightarrow{t\to\infty} \infty$

Intuition

15. $\sum_{n=1}^{\infty} \frac{\sqrt{n}+4}{n^2} \approx \sum \frac{1}{n^{3/2}}$

p-test attempt

$\frac{\sqrt{n}+4}{n^2} = \frac{\sqrt{n}}{n^2} + \frac{4}{n^2}$

$= \frac{1}{n^{3/2}} + \frac{4}{n^2}$

$= a_n + b_n$

$\because \sum a_n$ & $\sum b_n$ converge, by p-test, so $\sum(a_n + b_n)$ converges

Intuition

16. $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1} \approx \sum \frac{1}{n}$ fails p-test, but we don't know this is the case, at not formally. Can't back it up this way.

Our only concrete method is

$\frac{1}{3}\int_1^t \frac{3x^2\,dx}{x^3+1} = \frac{1}{3}\int_*^* \frac{du}{u} = \frac{1}{3}\ln|x^3+1|\Big]_1^t$

$u = x^3+1$

$du = 3x^2\,dx$

$= \frac{1}{3}\ln|t^3+1| - \frac{1}{3}\ln|2| \xrightarrow{t\to\infty} \infty$

$\boxed{\text{Diverges}}$

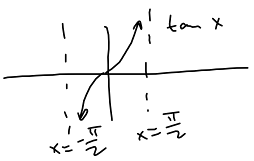
17. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

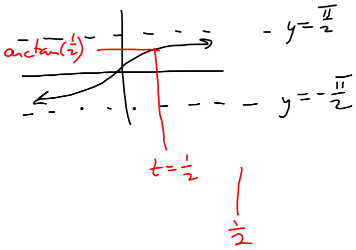
18. $\displaystyle\sum_{n=3}^{\infty} \frac{3n-4}{n^2-2n}$

$$\int \frac{du}{u^2+a^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$$

$$\int_1^t \frac{dx}{x^2+4} = \frac{1}{2}\arctan\left(\frac{x}{2}\right)\Big]_1^t$$

$$\frac{1}{2}\arctan\left(\frac{t}{2}\right) - \frac{1}{2}\arctan\left(\frac{1}{2}\right) \xrightarrow{t\to\infty} \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}(\text{number}) \text{ converges!}$$





19. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

22. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

18. $\sum_{n=3}^{\infty} \frac{3n-4}{n^2-2n} = \sum_{n=3}^{\infty} \left[\frac{2}{n} + \frac{1}{n-2}\right]$

$\frac{3n}{n^2-2n} = \frac{3}{n-2}$ $\frac{-4}{n^2-2n}$

Intuition:

$\frac{n\left[3-\frac{4}{n}\right]}{n[n-2]} = \frac{3-\frac{4}{n}}{n-2} \approx \frac{3}{n-2}$ when $n \to$ Big

diverges.

Integral Test is best we have & it's a BEAR

$$\int_3^t \frac{3x-4}{x^2-2x}\,dx = \int_3^t \left(\frac{2}{x} + \frac{1}{x-2}\right)dx = \Big[2\ln x + \ln(x-2)\Big]_3^t$$

$$= \Big[\ln(x^2) + \ln(x-2)\Big]_3^t$$

$a_n = 1$
$b_n = -1$

$\sum a_n, \sum b_n$ diverge,
$\sum [a_n + b_n] = 0$

$$= \ln(x^2(x-2))\Big|_3^t = \ln(t^2(t-2)) - \ln(9(3-2))$$

$\xrightarrow{t \to \infty} \infty - \ln 9 = \infty$

Diverges.

19. $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ **22.** $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

19. $\sum_{n=1}^{\infty} \boxed{\frac{\ln n}{n^3}}$ $\ln n < n$ $< \frac{1}{n^2}$

$$-\frac{1}{2}\frac{\ln(x)}{x^2} - \frac{1}{4x^2}$$

$$\int \frac{\ln x}{x^3}dx = uv - \int v\,du = -\frac{1}{2}x^{-2}\ln x - \int \left(-\frac{1}{2}x^{-2}\right)(x^{-1})dx$$

$u = \ln x \Rightarrow du = \frac{1}{x}dx$

$dv = x^{-3}dx \Rightarrow v = -\frac{1}{2}x^{-2}$

$$= -\frac{1}{2x^2}\ln x + \frac{1}{2}\int x^{-3}dx$$

$$= -\frac{\ln x}{2x^2} + \frac{1}{2}\left[-\frac{1}{2}x^{-2}\right] + C$$

$$= -\frac{\ln x}{2x^2} - \frac{1}{4}x^{-2} + C$$

$$\int_1^t \frac{\ln x}{x^3}dx$$

$$= \left[-\frac{\ln x}{2x^2} - \frac{1}{4x^2}\right]_1^t = -\frac{\ln t}{2t^2} - \frac{1}{4t^2} - \left[-\frac{\ln 1}{2} - \frac{1}{4}\right]$$

$$\xrightarrow{t\to\infty} 0 + 0 + \frac{1}{4} = \frac{1}{4}$$

converges.

22. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

$$\int \frac{dx}{x(\ln x)^2} = \int (\ln x)^{-2}\cdot\frac{1}{x}dx = \int u^{-2}du = \frac{u^{-1}}{-1} + C = -\frac{1}{\ln x} + C$$

$u = \ln x \Rightarrow$

$du = \frac{1}{x}dx$

$$\int_2^t \frac{1}{x(\ln x)^2}dx = \left[-\frac{1}{\ln x}\right]_2^t$$

$$= -\frac{1}{\ln t} - \left(-\frac{1}{\ln 2}\right) \xrightarrow{t\to\infty} \boxed{\frac{1}{\ln 2}}$$

So converges

27–28 Explain why the Integral Test can't be used to determine whether the series is convergent.

27. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos \pi n}{\sqrt{n}}$ $\cos \pi n \rightarrow (-1)^n$ **28.** $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{1 + n^2}$

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

OR on $[2,\infty)$
OR $[100,\infty)$
i.e., eventually

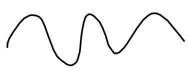
(a) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

$x = \frac{545\pi}{4}$, $x = \frac{547\pi}{4}$

27 $f'(x) = \dfrac{2\pi x \sin(\pi x) - \cos(\pi x)}{2x^{3/2}}$

Alternates signs



28 $f'(x) = -\dfrac{2(x^2+1)\cos x \sin x - 2x\cos^2 x}{(x^2+1)^2}$

changes signs infinitely often.

29–32 Find the values of p for which the series is convergent.

29. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

$$\int \frac{dx}{x(\ln x)^p} = \int (\ln x)^{-p} \cdot \frac{1}{x}\,dx = \int u^{-p}\,du = \frac{u^{-p+1}}{1-p} + C = \frac{1}{1-p}(\ln(x))^{1-p} + C$$

$$\int_2^t (\ln x)^{-p} \cdot \frac{1}{x}\,dx = \frac{1}{1-p}(\ln(x))^{1-p}\Big]_2^t$$

$$= \frac{1}{1-p}(\ln t)^{1-p} - \frac{1}{1-p}(\ln 2)^{1-p}$$

$\boxed{p \neq 1}$

Examine the $\lim_{t\to\infty}$

If power is positive, $(\ln t)^{\text{power}} \xrightarrow{t\to\infty} \infty$

Need $1-p \le 0$

$\boxed{1 \le p} \Rightarrow p > 1$

34. Leonhard Euler was able to calculate the exact sum of the p-series with $p = 2$:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

(See page 739.) Use this fact to find the sum of each series.

(a) $\sum_{n=2}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} - 1$

(b) $\sum_{n=3}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=4}^{\infty} \frac{1}{n^2}$

$= \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{3} \frac{1}{n^2}$

$= \frac{\pi^2}{6} - \frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2}$

$= \frac{\pi^2}{6} - 1 - \frac{1}{4} - \frac{1}{9}$

$= \frac{6\pi^2 - 36 - 9 - 4}{36} = \frac{6\pi^2 - 49}{36}$

(c) $\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \sum \frac{1}{2^2 n^2} = \sum \frac{1}{4} \cdot \frac{1}{n^2}$

$= \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{24}$

35. Euler also found the sum of the p-series with $p = 4$:

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Use Euler's result to find the sum of the series.

(a) $\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^4$

(b) $\sum_{k=5}^{\infty} \frac{1}{(k-2)^4}$

38. Find the sum of the series $\sum_{n=1}^{\infty} 1/n^5$ correct to three decimal places.

$$\int_{n+1}^{\infty} \le R_n \le \int_{n}^{\infty}$$

Want $n \ni R_n < .0005$

$$\int_n^t \frac{dx}{x^5} = \int_n^t x^{-5}dx = \left.\frac{x^{-4}}{-4}\right]_n^t = -\frac{1}{4}\left[t^{-4} - n^{-4}\right] \xrightarrow{t \to \infty} \left(-\frac{1}{4}\right)\left(-n^{-4}\right)$$

$$= \frac{1}{4n^4} \underset{\text{Want}}{<} .0005$$

$$1 < .002n^4$$

$$500 = \frac{1}{.002} < n^4$$

$$4.728708045 \approx \sqrt[4]{500} < \sqrt[4]{n^4} = |n| = n$$

Let $n = 5$. Should do it!

$$\frac{1}{1^5} + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} \approx 1.036661789$$

Maple thinks the actual value is approximately 1.036927755