11 Infinite Sequences and Series

§11.2 #s 1-4, 5-8, 9, 12, 15, 16, 20, 21, 24, 26-30, 33, 43, 44

↳ See my spreadsheet. Run those sums out to 20 terms!

π = 3.14159 26535 89793 23846 26433 83279 50288 . . .

$$\pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \cdots$$

11.1 Sequences $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3 \ldots\}$

11.2 Series $a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n$

Put a +-sign between the terms of a sequence!

$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots$

Johann Friedrich Gauss

$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots + 100$

$101 \cdot 50 = 5050$

$(n+1)\frac{n}{2} = \frac{n(n+1)}{2} = \sum_{k=1}^{n} k = 1+2+\cdots+n$

Sequence of partial sums:

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3$

$s_4 = a_1 + a_2 + a_3 + a_4$

$\vdots$

$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$

These partial sums form a new sequence $\{s_n\}$, which may or may not have a limit.

If $\lim_{n\to\infty} s_n = s$ exists (as a finite number), then, as in the preceding example, we call it the sum of the infinite series Σa_n.

2 Definition Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series Σa_n is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

An important example of an infinite series is the **geometric series**

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

$$= \sum_{n=0}^{\infty} ar^n$$

annuities

immigration & exponential growth

Each term is obtained from the preceding one by multiplying it by the **common ratio** r.

$$S'_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

$$-\quad rS'_n = ar + ar^2 + a^3 + ar^4 + \cdots + ar^{n-1} + ar^n$$

$$S'_n - rS'_n = a - ar^n$$

$$S'_n(1-r) = a - ar^n$$

$$S' = \frac{a(1-r^n)}{1-r} \xrightarrow{n \to \infty} \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \nexists & \text{if } |r| \geq 1 \end{cases}$$

Telescoping Sum

If $r = 1$, then $s_n = \underbrace{a + a + \cdots + a}_{n \text{ of 'em}} = na \xrightarrow{n \to \infty} \cancel{\pm\infty}$. ∞

If $-1 < r < 1$, we know that as $r^n \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{a}{1-r}\lim_{n\to\infty} r^n = \frac{a}{1-r}$$

Thus when $|r| < 1$ the geometric series is convergent and its sum is $a/(1 - r)$.

If $r \leq -1$ or $r > 1$, the sequence $\{r^n\}$ is divergent and so, by Equation 3, $\lim_{n\to\infty} s_n$ does not exist.

3 $$\boxed{s_n = \frac{a(1-r^n)}{1-r}}$$

4 The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geq 1$, the geometric series is divergent.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots+\frac{1}{2^n}+\cdots = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

$$\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^n}+\cdots=1$$

$$\sum_{n=1}^{\infty} ar^{n-1} =$$

$$r = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{8}\cdot\frac{4}{1} = \frac{1}{2} = \text{common ratio.}$$

$a = 1/2$ = 1st term

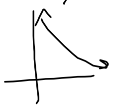
$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{1}{2}+\frac{1}{4}+\cdots$$

$$= ar^0 + ar^1 + \cdots$$

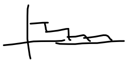
n	Sum of first n terms
	S_n
1	0.50000000
2	0.75000000
3	0.87500000
4	0.93750000
5	0.96875000
6	0.98437500
7	0.99218750
10	0.99902344
15	0.99996948
20	0.99999905
25	0.99999997

Recall the *p*-test for improper integrals? I'll never ask *you* to make this argument, but you should try to follow it. The Harmonic Series is the *p* = 1 case for infinite series, just the same as 1/*x* is the *p* = 1 case for

$\int \qquad \int_1^\infty \frac{dx}{x^p} \longleftrightarrow \sum_{n=1}^\infty \frac{1}{n^p}$.



$\sum$



$\int_1^\infty$ converges for $p>1$ & diverges for $p \le 1$. Same deal with series! So you already know more about these infinite series than you thought!

Show that the **harmonic series**

$$\sum_{n=1}^\infty \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent. $p=1$ Recall $\int_1^\infty \frac{dx}{x}$ Diverges to $+\infty$. Well, so does $\sum_{n=1}^\infty \frac{1}{n}$!

$\int_1^\infty \frac{dx}{x}$ diverges

so

$\sum_{n=1}^\infty \frac{1}{n}$ diverges

Harmonic Series

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

$S_1' = 1 \quad = 1 + \frac{0}{2} \quad = a_0$

$S_2' = 1 + \frac{1}{2} \qquad S_{2^1}' = a_1$

$S_4' = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2} \qquad S_{2^2}' > a_2$

$S_8' = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{2}{2} + \frac{1}{2} = 1 + \frac{3}{2} \qquad S_{2^3}' > a_3$

$S_{16}' = 1 + \cdots + \frac{1}{16} > \frac{3}{2} + \frac{2}{2} + \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right) > 1 + \frac{3}{2} + \frac{1}{2} = 1 + \frac{4}{2} \qquad S_{2^4}' > a_4$

Consider $a_1, a_2, a_3, a_4, \ldots$

$1, \ 1 + \frac{1}{2}, \ 1 + \frac{3}{2}, \ 1 + \frac{4}{2}$

$a_n = 1 + \frac{n}{2} < S_n'$

$\sum a_n = \sum_{k=1}^n \left(1 + \frac{k}{2}\right) = \sum_{k=1}^n 1 + \frac{1}{2}\sum_{k=1}^n k$

$= n + \frac{1}{2}\left(\frac{n(n+1)}{2}\right)$

$\xrightarrow{n \to \infty} \infty$ Diverges,

and $S_n' > a_n$

$S_n' \xrightarrow{n \to \infty} \infty$

The most Basic Divergence Criterion

6 Theorem If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

$\sum a_n$ converges $\Longrightarrow$ $\lim a_n = 0$

Converse doesn't hold.

$\lim a_n = 0 \Longrightarrow \sum a_n$ converges does not hold

Counterexample : $\sum \frac{1}{n}$ Harmonic Series Diverges

in spite of $\frac{1}{n} \xrightarrow{n\to\infty} 0$

7 The Test for Divergence If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Is Contrapositive

Not $\lim a_n = 0 \Longrightarrow$ Not convergent

$a_n \not\to 0 \Longrightarrow \sum a_n$ diverges

8 Theorem If $\Sigma\, a_n$ and $\Sigma\, b_n$ are convergent series, then so are the series $\Sigma\, ca_n$ (where c is a constant), $\Sigma\,(a_n + b_n)$, and $\Sigma\,(a_n - b_n)$, and

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

$3(1+2+3+4) = 3\cdot1 + 3\cdot2 + 3\cdot3 + 3\cdot4$

(ii) $\displaystyle\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

$\sum\left(\left(\frac{1}{2}\right)^n + \frac{1}{n^2}\right) = \sum\left(\frac{1}{2}\right)^n + \sum\left(\frac{1}{n^2}\right)$

$\sum\left(\left(\frac{1}{2}\right)^n - \frac{1}{n^2}\right) = \sum\left(\frac{1}{2}\right)^n - \sum\left(\frac{1}{n^2}\right)$

(iii) $\displaystyle\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

1. (a) What is the difference between a sequence and a series?
(b) What is a convergent series? What is a divergent series?

2. Explain what it means to say that $\Sigma_{n=1}^{\infty} a_n = 5$.

3–4 Calculate the sum of the series $\sum_{n=1}^{\infty} a_n$ whose partial sums are given.

3. $s_n = 2 - 3(0.8)^n$

Rarely be handed closed-form expression for S_n.

$2 - 3(0.8)^n \xrightarrow{n \to \infty} 2$

$\to 0$

4. $s_n = \dfrac{n^2 + 1}{4n^2 + 1} \xrightarrow{n \to \infty} \dfrac{1}{4}$

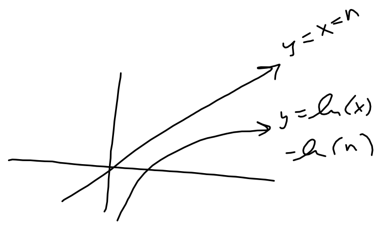
5–8 Calculate the first eight terms of the sequence of partial sums correct to four decimal places. Does it appear that the series is convergent or divergent?

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ See video & Spreadsheet!

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)} \geq \sum_{n=1}^{\infty} \frac{1}{n}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{1+\sqrt{n}}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$



9–14 Find at least 10 partial sums of the series. Graph both the sequence of terms and the sequence of partial sums on the same screen. Does it appear that the series is convergent or divergent? If it is convergent, find the sum. If it is divergent, explain why.

9. $\sum_{n=1}^{\infty} \frac{12}{(-5)^n}$

Video & Spreadsheet

12. $\sum_{n=1}^{\infty} \frac{7^{n+1}}{10^n}$

12 $\sum \frac{7^n \cdot 7^1}{10^n} = \sum 7\left(\frac{7}{10}\right)^n = \frac{7}{1-\frac{7}{10}} = \frac{70}{3}$

15. Let $a_n = \dfrac{2n}{3n + 1}$.

(a) Determine whether $\{a_n\}$ is convergent.

(b) Determine whether $\Sigma_{n=1}^{\infty} a_n$ is convergent.

(a) $a_n \xrightarrow{n \to \infty} \frac{2}{3}$

(b) $0 \neq \frac{2}{3} \Rightarrow \sum a_n$ diverges

16. (a) Explain the difference between

$$\sum_{i=1}^{n} a_i \quad \cancel{\text{and}} = \quad \sum_{j=1}^{n} a_j$$

is the difference between index/dummy var is different, $\int_0^1 x\,dx$ & $\int_0^1 t\,dt$ is all

(b) Explain the difference between

$$\sum_{i=1}^{n} a_i \quad \text{and} \quad \sum_{i=1}^{n} a_j$$

$a_1 + a_2 + \cdots + a_n$ and $a_j + a_j + \cdots + a_j = n\,a_j$

17–26 Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

20. $2 + 0.5 + 0.125 + 0.03125 + \cdots$

Find a & r

$\frac{.5}{2} = \frac{\frac{1}{2}}{2} = \frac{1}{4} = r$

$a = a_1 = 2$

$\frac{.125}{.5} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4} = r < 1$

$S = \frac{2}{1-r} = \frac{2}{1-\frac{1}{4}} = \frac{2}{\frac{3}{4}} = \frac{2}{1} \cdot \frac{4}{3} = \boxed{\frac{8}{3}}$

21. $\sum_{n=1}^{\infty} 6(0.9)^{n-1}$

$e \approx 2.72$

26. $\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}} = \sum \frac{e^n}{3^n \cdot 3^{-1}} = \sum \frac{e^{n-1} e^1}{3^{n-1}} = \sum e^1 \frac{e^{n-1}}{3^{n-1}} = \sum e^1 \left(\frac{e}{3}\right)^{n-1} = \frac{e}{1-\frac{e}{3}} = \frac{e}{\frac{3-e}{3}} = \frac{3e}{3-e}$

24. $\sum_{n=0}^{\infty} \frac{1}{(\sqrt{2})^n} = \sum \left(\frac{1}{\sqrt{2}}\right)^n = \sum \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}\right)^{n-1}$

$a = e$

$-1 < r = \frac{e}{3} < 1$

$r = \frac{1}{\sqrt{2}}$

$a = \frac{1}{\sqrt{2}}$

$r = \frac{1}{\sqrt{2}}$ $S =$

27–42 Determine whether the series is convergent or divergent. If it is convergent, find its sum.

27. $\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\frac{1}{15}+\cdots$ $=\sum_{k=1}^{n}\frac{1}{3k} \;\; := \;\; \frac{1}{3}\sum_{k=1}^{n}\frac{1}{k} = \frac{1}{3}\cdot$ Harmonic diverges

28. $\frac{1}{3}+\frac{2}{9}+\frac{1}{27}+\frac{2}{81}+\frac{1}{243}+\frac{2}{729}+\cdots$

$S_6 = \frac{1}{3}+\frac{1}{9}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{81}+\frac{1}{243}+\frac{1}{729}+\frac{1}{729}$

$= \underbrace{\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots+\frac{1}{729}}_{\sum_{k=1}^{6}\frac{1}{3}\cdot\left(\frac{1}{3}\right)^{k-1}} + \frac{1}{9}+\frac{1}{81}+\frac{1}{729}$

$\quad + \sum_{k=1}^{3}\frac{1}{9}\left(\frac{1}{9}\right)^{k-1}$ Gives $\sum_{k=1}^{\infty}\frac{1}{3}\left(\frac{1}{3}\right)^{k-1}+\sum_{k=1}^{\infty}\frac{1}{9}\left(\frac{1}{9}\right)^{k-1}$

$= \frac{\frac{1}{3}}{1-\frac{1}{3}} + \frac{\frac{1}{9}}{1-\frac{1}{9}} = \frac{\frac{1}{3}}{\frac{2}{3}} + \frac{\frac{1}{9}}{\frac{8}{9}}$

$= \frac{1}{3}\cdot\frac{3}{2}+\frac{1}{9}\cdot\frac{9}{8}$

$= \frac{1}{2}+\frac{1}{8}=\frac{5}{8}$

29. $\sum_{n=1}^{\infty}\frac{n-1}{3n-1}$ diverges $a_n=\frac{n-1}{3n-1}\xrightarrow{n\to\infty}\frac{1}{3}\neq 0$

30. $\sum_{k=1}^{\infty}\frac{k(k+2)}{(k+3)^2}$ $a_n=\frac{k^2+2k}{k^2+6k+9}\xrightarrow{n\to\infty} 1$

31. $\sum_{n=1}^{\infty}\frac{1+2^n}{3^n}$

33. $\sum_{n=1}^{\infty}\sqrt[n]{2}$ $.3^n = .3\cdot .3^{n-1}$

34. $\sum_{n=1}^{\infty}\left[(0.8)^{n-1}-(0.3)^n\right]$

43–48 Determine whether the series is convergent or divergent by expressing s_n as a telescoping sum (as in Example 7). If it is convergent, find its sum.

43. $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$ Partial Fractions

44. $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$

Props of Logs-

$\ln\left(\frac{n}{n+1}\right) = \ln(n) - \ln(n+1)$

$\frac{2}{n^2-1} = \frac{1}{n-1} - \frac{1}{n+1}$

So $S_n = \frac{1}{2-1} - \frac{1}{2+1} + \frac{1}{3-1} - \frac{1}{3+1} + \frac{1}{4-1} - \frac{1}{4+1} \ldots + \frac{1}{n-1} - \frac{1}{n+1}$

$= \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} + \ldots + \frac{1}{n-2} - \frac{1}{n} + \frac{1}{n-1} - \frac{1}{n+1}$

$= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \; \boxed{\xrightarrow{n \to \infty} \frac{3}{2}}$

$S_6 = \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} + \frac{1}{6} - \frac{1}{8} +$

2 3 4 5 6 7

$-\frac{1}{n+1} \quad -\frac{1}{n+2}$