11.1 Sequences

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

The number a_1 is called the *first term*, a_2 is the *second term*, and in general a_n is the *nth term*. We will deal exclusively with infinite sequences and so each term a_n will have a successor a_{n+1}.

Notice that for every positive integer n there is a corresponding number a_n and so a sequence can be defined as a function whose domain is the set of positive integers.

But we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n.

Notation: The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

$\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}_{n\in\mathbb{N}}$

(a) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ $a_n = \frac{n}{n+1}$

$\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$

(b) $\{-\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \ldots\}$

$a_n =$

$n=1$ $\quad \frac{2}{3} = \left(\frac{n+1}{n+2}\right)(-1)^1$

$= \left(\frac{n+1}{n+2}\right)(-1)^n$

$n=2$ $\quad \left(\frac{2+1}{2+2}\right)(-1)^n$

$a_n = (-1)^n \frac{n+1}{3^n}$

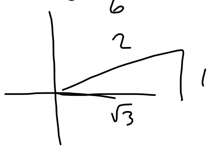
(c) $\{\sqrt{n-3}\}_{n=3}^{\infty}$ $a_n = \sqrt{n-3},\ n \geqslant 3$

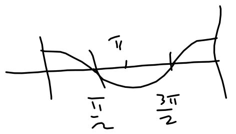
$0, 1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}$

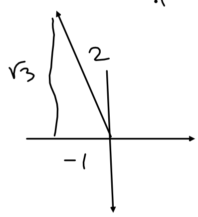
(d) $\left\{\cos\frac{n\pi}{6}\right\}_{n=0}^{\infty}$ $a_n = \cos\frac{n\pi}{6},\ n \geqslant 0$

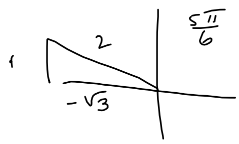
$1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -1, -\frac{\sqrt{3}}{2}, -\frac{1}{2}, \ldots$

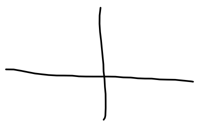
$\cos\frac{\pi}{6}$ $\cos(2\frac{\pi}{6}) = \cos(\frac{\pi}{3})$, $\cos\frac{\pi}{2}$, $\cos(\frac{2\pi}{3})$











A sequence such as the one in Example 1(a), $a_n = n/(n + 1)$, can be pictured either by plotting its terms on a number line, as in Figure 1, or by plotting its graph, as in Figure 2.

Rarely

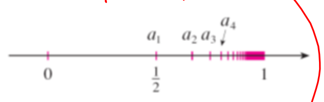


Figure 1

Yeah

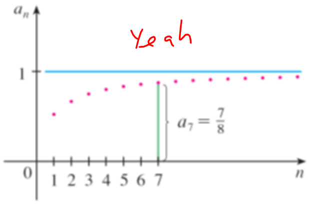
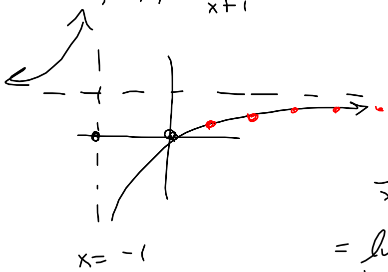


Figure 2

$f(x) = \frac{x}{x+1}$ informs this



$y = 1$ from

$\lim_{x \to \infty} \left(\frac{x}{x+1}\right)$

$= \lim_{n \to \infty} \frac{n}{n+1} = 1$

1 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty \quad \text{OR} \quad a_n \xrightarrow{n\to\infty} L$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

EVENTUALLY $|a_n - L| <$ Small

2 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if} \quad n > N \quad \text{then} \quad |a_n - L| < \varepsilon$$

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$$

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

~~Proof~~

Scratch want $\left|1 - \frac{n}{n+1}\right| < \varepsilon$

$$\Rightarrow \left|\frac{n+1-n}{n+1}\right| = \left|\frac{1}{n+1}\right| = \frac{1}{n+1} < \varepsilon$$

$$\Rightarrow 1 < \varepsilon(n+1) \quad (n+1 > 0)$$

$$1 < n\varepsilon + \varepsilon$$

$$n\varepsilon + \varepsilon > 1$$

$$n\varepsilon > 1 - \varepsilon$$

$$n > \frac{1-\varepsilon}{\varepsilon}$$

Proof

Let $\varepsilon > 0$ be given.

Assume $\varepsilon < 1$, also.

Define $N \equiv \frac{1-\varepsilon}{\varepsilon}$. Then for every $n > N$,

$$\left|1 - \frac{n}{n+1}\right| = \left|\frac{n+1-n}{n+1}\right| = \left|\frac{1}{n+1}\right|$$

$$< \frac{1}{\frac{1-\varepsilon}{\varepsilon} + 1} = \frac{1}{\frac{1-\varepsilon+\varepsilon}{\varepsilon}} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon \quad \blacksquare$$

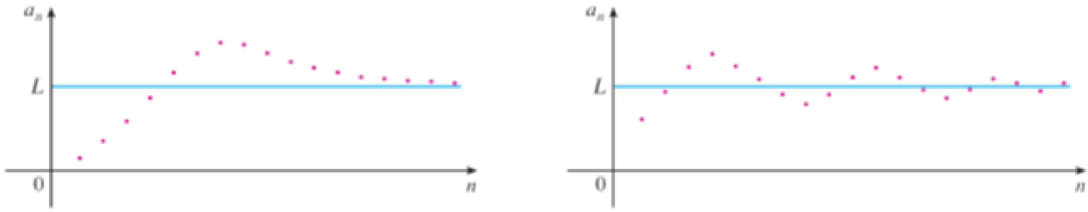


Figure 3

Graphs of two sequences with $\lim_{n\to\infty} a_n = L$

The ε-tube.

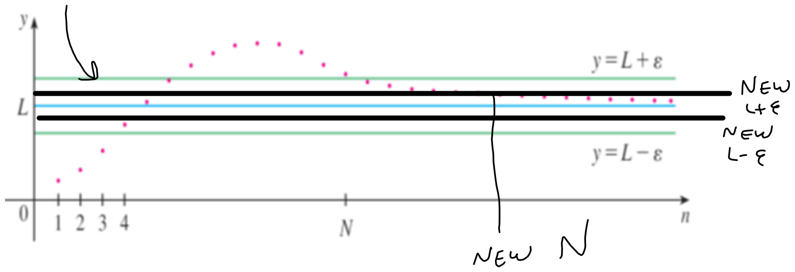


Figure 5

"Eventually" means there is an $N > 0$ such that

$|a_n - L| < \varepsilon$

(i.e. $L - \varepsilon < a_n < L + \varepsilon$)

yer in the tube, baby!

$\lim_{n\to\infty} a_n = L$ means

Given $\varepsilon > 0$, $\exists\, N > 0 \ni$

$|a_n - L| < \varepsilon \quad \forall\, n > N$

3 Theorem If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

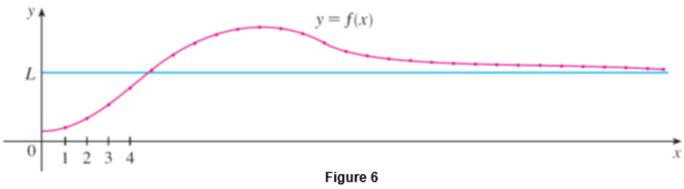


Figure 6

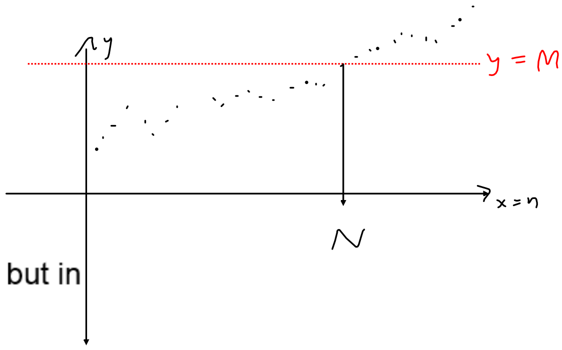
In particular, since we know that $\lim_{x\to\infty}(1/x^r) = 0$ when $r > 0$, we have

$$\boxed{4} \qquad \lim_{n\to\infty} \frac{1}{n^r} = 0 \qquad \text{if } r > 0$$

If a_n becomes large as n becomes large, we use the notation $\lim_{n\to\infty} a_n = \infty$. Consider the definition

5 Definition $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$\text{if} \quad n > N \quad \text{then} \quad a_n > M$$



If $\lim_{n\to\infty} a_n = \infty$, then the sequence $\{a_n\}$ is divergent but in a special way. We say that $\{a_n\}$ diverges to ∞.

Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

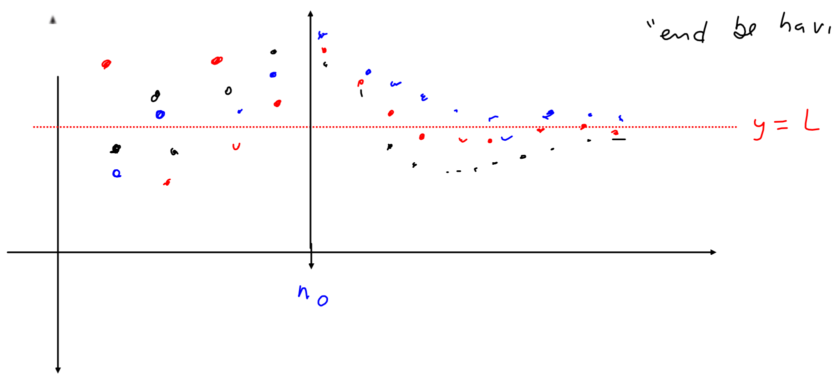
Be careful. Sometimes $\{a_n\}$ & $\{b_n\}$ don't converge, separately. BOLO for this.

The Squeeze Theorem can also be adapted for sequences as follows (see Figure 7).

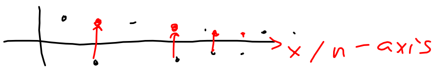
Squeeze Theorem for Sequences

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

$\lim_{n\to\infty} a_n$ is all about "eventually," "end behavior."



6 Theorem If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.



7 Theorem If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

F conts means

$$\lim_{n\to\infty} f(a_n) = f(L) \qquad \lim_{x\to a} f(x) = f(a)$$

9 The sequence $\{r^n\}$ is convergent if $-1 < r \leqslant 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

$r = .2$

$.2, .2^2, .2^3, .2^4, \ldots, .2^n, \ldots$

is monotone decreasing

Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant 1$. A sequence is **monotonic** if it is either increasing or decreasing.

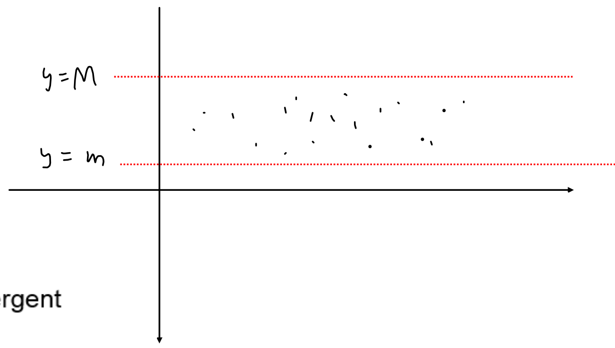
11 Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leqslant M \qquad \text{for all } n \geqslant 1$$

It is **bounded below** if there is a number m such that

$$m \leqslant a_n \qquad \text{for all } n \geqslant 1$$

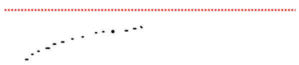
If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.



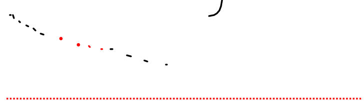
not every bounded sequence is convergent
$a_n = (-1)^n$

not every monotonic sequence is convergent ($a_n = n \to \infty$).

But if a sequence is both bounded *and* monotonic, then it must be convergent.



bdd above. & increasing
bdd below & decreasing



12 **Monotonic Sequence Theorem** Every bounded, monotonic sequence is convergent.

The proof of Theorem 12 is based on the **Completeness Axiom** for the set of ℝ real numbers, which says that if *S* is a nonempty set of real numbers that has an upper bound *M* ($x \le M$ for all *x* in *S*), then *S* has a **least upper bound** *b*.

Shortest basketball player.

(This means that *b* is an upper bound for *S*, but if *M* is any other upper bound, then $b \le M$.)

The Completeness Axiom is an expression of the fact that there is no gap or hole in the real number line.

Likewise if

$\emptyset \ne S'$ & S' has l.b.

Then S' has Greatest lower bound.

Tallest midget.

§11.1 #s 1, 2, 3, 6, 7, 10*, 13, 16, 18–28

Recursion

41, 42, 44, ~~4~~45, 47, 48

79, 80 Bonus

1. (a) What is a sequence?
(b) What does it mean to say that $\lim_{n\to\infty} a_n = 8$?
(c) What does it mean to say that $\lim_{n\to\infty} a_n = \infty$?

$f(x) = x^2$

$f(n) = a_n = n^2$

(a) A function, $f : \mathbb{N} \longrightarrow \mathbb{R}$

(A func from natural #s into real #s).

(b) $\lim_{n\to\infty} a_n = 8$ means,

Stayin' in the ε-tube, baby!

given $\varepsilon > 0$, there is an $N > 0$ such that

$|a_n - 8| < \varepsilon$ whenever $n > N$.

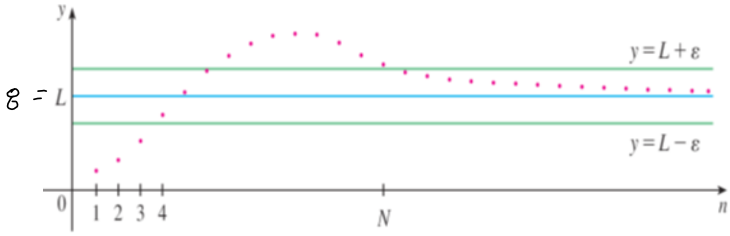


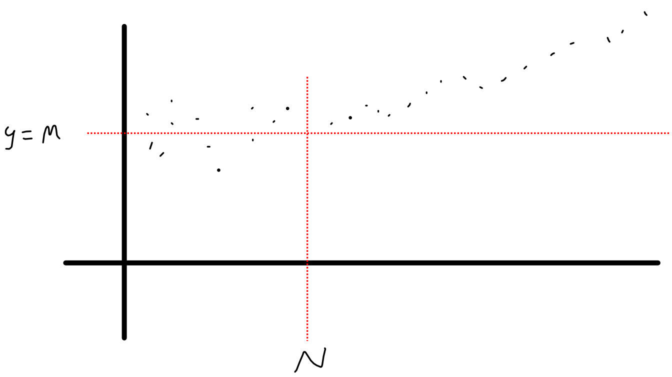
Figure 5

(c) $\lim_{n\to\infty} a_n = \infty$ means,

given $M > 0$, there is $N > 0$ such that

$a_n > M$ for all $n > N$.

we'll punch thru any ceiling you place above our heads



2. (a) What is a convergent sequence? Give two examples.
(b) What is a divergent sequence? Give two examples.

(a) A convergent sequence is a sequence that comes arbitrarily close to a value, L, and STAYS close.

$$a_n = \frac{n+3}{n-7} \xrightarrow{n \to \infty} 1 = L$$

$$a_n = \frac{1}{n} \xrightarrow{n \to \infty} 0 = L$$

(b) A divergent sequence is a sequence that's not convergent!

$$a_n = n \xrightarrow{n \to \infty} \infty \quad \text{(i.e., diverges!)}$$

$$a_n = (-1)^n n \xrightarrow{n \to \infty} \nexists$$

basically $\pm\infty$, depending on odds/evens.

$a_n = (-1)^n$

$\{-1, 1, -1, 1, \ldots\}$ diverges

Converges means, given $\varepsilon > 0$, $\exists\ N > 0 \ni |a_n - L| < \varepsilon \ \forall\ n > N$

Divergent includes

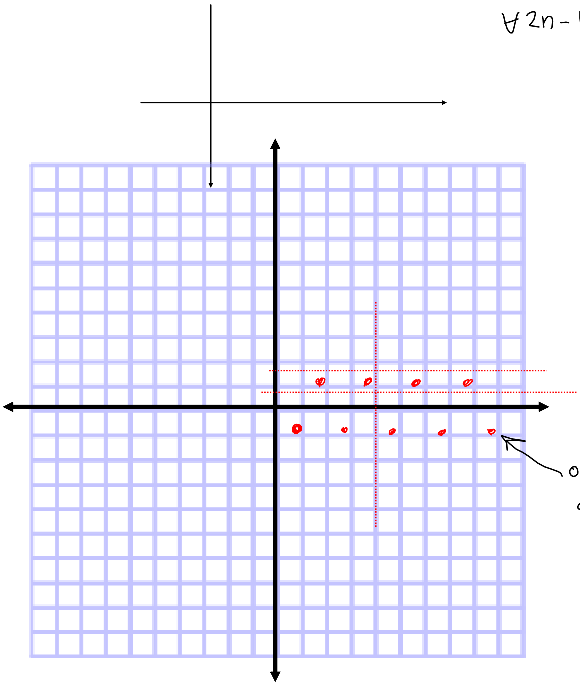
There's an $\varepsilon > 0$ such that $\nexists N \ni |a_n - L| < \varepsilon \ \forall\ n > N$

Let $\varepsilon = \frac{1}{2}$ $\quad (-1)^n$ doesn't converge to $L = 1$

Then $|a_n - L| = |-1 - L|$ OR $|1 - L|$

when n is odd, $|a_n - L| = |-1 - 1| = 2 > \varepsilon$

$\forall\ 2n-1$.



outside ε-tube and continues like this, forever, no matter how big N is

3–12 List the first five terms of the sequence.

3. $a_n = \dfrac{2n}{n^2 + 1}$

$\frac{2}{2}, \frac{4}{5}, \frac{6}{10}, \frac{8}{17}, \frac{10}{26}$

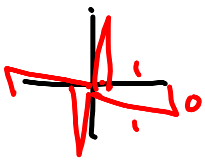
10. $a_1 = 6, \quad a_{n+1} = \dfrac{a_n}{n}$

6. $a_n = \cos\dfrac{n\pi}{2}$

7. $a_n = \dfrac{1}{(n+1)!}$

$0, -1, 0, 1, 0, -1$

Recursive Def'n!



$6, \frac{6}{2}, \frac{3}{3}, \frac{1}{4}, \frac{\frac{1}{4}}{5} = \frac{1}{20},$

$= 6, 3, 1, \frac{1}{4}, \frac{1}{20}, \frac{1}{120}$

13–18 Find a formula for the general term a_n of the sequence assuming that the pattern of the first few terms continues.

13. $\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots\}$ $\frac{1}{1}, \frac{1}{3}, \ldots$ $\frac{1}{2n-1}$

16. $\{5, 8, 11, 14, 17, \ldots\}$

Add 3... start @ 5 : $5+3n,\ n \geq 0$

(start @ $0 = n$)

start @ $n=1$

$5+0,\ 5+3,\ 5+2\cdot 3, \ldots$

$5+3(n-1)$

start @ $n=17$

$5+3(n-17)$

18. $\{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$

$\boxed{\sin\left(\frac{n\pi}{2}\right)\ ?}$ Yup

w/o #6, I'd've been thinking

6. $a_n = \cos\frac{n\pi}{2}$

$$\begin{cases} 0 & \text{if } n = 2K \quad \text{Evens} \\ (-1)^{K-1} & \text{if } n = 2K-1 \quad \text{odds} \end{cases}$$

Even - odds

odd - evens

$n=1$ $(-1)^{n-1} = 1,$

$1 = 2(1) - 1 \implies K = 1$

$(-1)^{1-1} = -1^0 = 1$

$n=3$ $(-1)^{3-1} = (-1)^2 = 1$

$n = 3 = 2(2) - 1 \rightarrow K = 2$

$(-1)^{2-1} = (-1)^1 = -1$

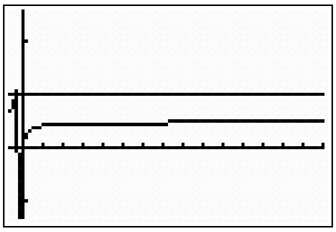
19–22 Calculate, to four decimal places, the first ten terms of the sequence and use them to plot the graph of the sequence by hand. Does the sequence appear to have a limit? If so, calculate it. If not, explain why.

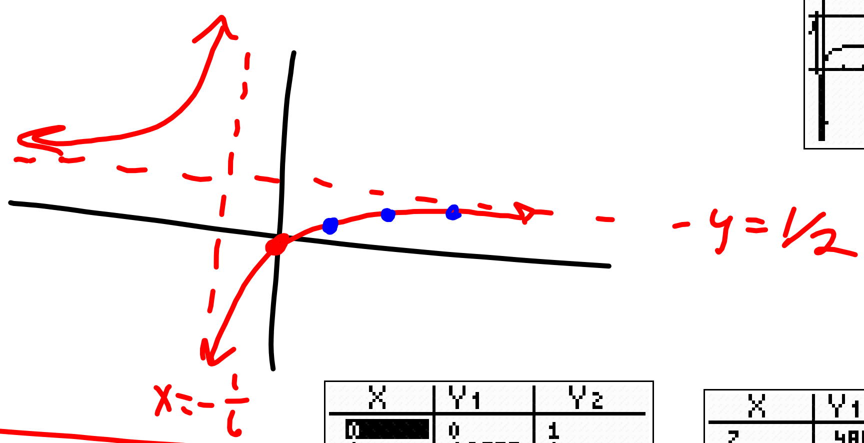
!?

19. $a_n = \dfrac{3n}{1 + 6n}$ **20.** $a_n = 2 + \dfrac{(-1)^n}{n}$

Heck wit' 'dis.

$$f(x) = \frac{3x}{6x+1} = \frac{3x}{6(x+\frac{1}{6})} = \frac{x}{2(x+\frac{1}{6})}$$





X	Y1	Y2
0	0	1
1	.42857	1
2	.46154	1
3	.47368	1
4	.48	1
5	.48387	1
6	.48649	1

X=0

X	Y1	Y2
7	.48837	1
8	.4898	1
9	.49091	1
10	.4918	1
11	.49254	1
12	.49315	1
13	.49367	1

X=13

Yes, it has a limit, $L = \frac{1}{2}$

Plot1 Plot2 Plot3
\Y1 = X/(2(X+1/6))
\Y2 = 1
\Y3 =
\Y4 =
\Y5 =
\Y6 =
\Y7 =

$$\lim_{n\to\infty} \frac{3n}{6n+1} =$$

$$\lim_{n\to\infty} \frac{3n}{n(6+\frac{1}{n})}$$

$$= \lim_{n\to\infty} \frac{3}{6+\frac{1}{n}} = \frac{3}{6} = \boxed{\frac{1}{2} = L}$$

$\frac{1}{n} \to 0$

21. $a_n = 1 + \left(-\frac{1}{2}\right)^n$

22. $a_n = 1 + \dfrac{10^n}{9^n}$

$Y_1 = 1 + ((-)\frac{1}{2})^X$

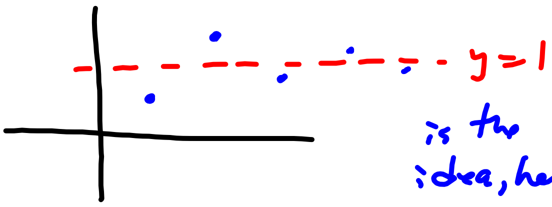
X	Y1	Y2
1	.5	1
2	1.25	1
3	.875	1
4	1.0625	1
5	.96875	1
6	1.0156	1
7	.99219	1

X=7

X	Y1	Y2
9	.99805	1
10	1.001	1
11	.99951	1
12	1.0002	1
13	.99988	1
14	1.0001	1
15	.99997	1

X=15

$\lim_{n\to\infty} a_n = 1$



is the idea, here

$= 1 + \left(\frac{10}{9}\right)^n$

$\left|\frac{10}{9}\right| > 1 \Rightarrow$

Diverges!

$-1 < r \le 1$

r^n converges, so does $1 + r^n$

a_n converges & b_n doesn't, then $a_n + b_n = c_n$ diverges.

a_n, b_n converge $\Rightarrow a_n + b_n$ converges

a_n, b_n diverge $\Rightarrow a_n + b_n$ uncertain.

$a_n = (-1)^n$ Diverge, but

$b_n = (-1)^{n+1}$

$a_n + b_n = (-1)^n + (-1)^{n+1}$

$= 0$ converges

$0, 0, 0, 0, \ldots$

23–56 Determine whether the sequence converges or diverges. If it converges, find the limit.

$-1 < r = .2 \le 1$

23. $a_n = 1 - (0.2)^n \xrightarrow{n\to\infty} 0$

24. $a_n = \dfrac{n^3}{n^3 + 1} \xrightarrow{n\to\infty} 1$

Converges

$\lim_{n\to\infty} (1 - .2^n)$

$= \lim_{n\to\infty} (1) - \lim_{n\to\infty} (.2^n)$

Can do this b/c $1, .2^n$ both converge, so $\lim(a_n - b_n) = \lim a_n - \lim b_n$

$= 1 - 0 = \boxed{1 = L}$

25. $a_n = \dfrac{3 + 5n^2}{n + n^2} \xrightarrow{n \to \infty} 5$

L'Hôpital
oR
College Algebra
looking for HoR. AS.
OR some teach
this method:

$$\frac{3+5n^2}{n+n^2} = \frac{n^2\left(\frac{3}{n^2}+5\right)}{n^2\left(\frac{1}{n}+1\right)} = \frac{\frac{3}{n^2}+5}{\frac{1}{n}+1} \xrightarrow{n\to\infty} \frac{0+5}{0+1} = 5$$

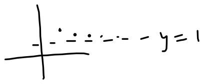
26. $a_n = \dfrac{n^3}{n + 1}$

$$\frac{n}{n^3+1} \xrightarrow{n\to\infty} 0$$

$$\lim_{n\to\infty} a_n \nexists$$

Grows w/o bound.

$e, e^{\frac{1}{2}} = \sqrt{e}, e^{\frac{1}{3}} = \sqrt[3]{e}$



27. $a_n = e^{1/n}$

$$\lim_{n\to\infty} a_n = y = \lim_{n\to\infty} e^{\frac{1}{n}}$$

$$\ln y = \lim_{n\to\infty} \frac{1}{n} = 0$$

$$e^{\ln y} = e^0$$

$$\boxed{y = 1 = \lim_{n\to\infty} e^{\frac{1}{n}}}$$

$0<x<1$: $x^n < x$ $(n>1)$

$-1<x<0$: $|x^n| < |x|$

28. $a_n = \dfrac{3^{n+2}}{5^n} = \dfrac{3^2 \cdot 3^n}{5^n}$

$$= 9 \cdot \left(\frac{3}{5}\right)^n$$

$$\xrightarrow{n\to\infty} 0,$$

b/c $\left|\frac{3}{5}\right| < 1$

41. $\{n^2e^{-n}\}$

L'H: $\dfrac{\frac{d}{dn}[\text{top}]}{\frac{d}{dn}[\text{bottom}]}$

$$\frac{n^2}{e^n} \xrightarrow[\text{L'H}]{n\to\infty} \frac{2n}{e^n}$$

$$\xrightarrow[\text{L'H}]{n\to\infty} \frac{2}{3^n} \xrightarrow{n\to\infty} 0$$

42. $a_n = \ln(n+1) - \ln n$

$$= \ln\left(\frac{n+1}{n}\right)$$

$$\xrightarrow{n\to\infty} \ln(1)$$

$$= 0.$$

limit can pass inside a continuous function

$$\lim_{n\to\infty} \left(\ln\left(\frac{n+1}{n}\right)\right)$$

$$= \ln\left(\lim_{n\to\infty}\left(\frac{n+1}{n}\right)\right)$$

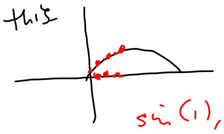
$$= \ln(1) = 0$$

45. $a_n = n \sin(1/n)$

$-1 \le \sin\left(\frac{1}{n}\right) \le 1$

$-n \le n \sin\left(\frac{1}{n}\right) \le n$

Don't see squeeze for this



$\sin(1), \sin\left(\frac{1}{2}\right), \sin\left(\frac{1}{3}\right)$ ... $\sin\left(\frac{1}{100}\right), \ldots$

$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \infty \cdot 0$

$= \frac{0}{\frac{1}{\infty}} = \frac{0}{0}$

$n \sin\left(\frac{1}{n}\right) =$

$\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \xrightarrow[L'H]{n\to\infty} \frac{-\frac{1}{n^2}\cos\left(\frac{1}{n}\right)}{-\frac{1}{n^2}} = \cos\left(\frac{1}{n}\right) \xrightarrow{n\to\infty} 1$

$\frac{d}{dn}\left[\frac{1}{n}\right] = \frac{d}{dn}\left[n^{-1}\right] = -1 \cdot n^{-2} = -\frac{1}{n^2}$

44. $a_n = \sqrt[n]{2^{1+3n}}$

$2^{3n} = (2^3)^n$

$= \sqrt[n]{2^1 \cdot 2^{3n}}$

$= \sqrt[n]{2}\,\sqrt[n]{2^{3n}}$

$= \sqrt[n]{2}\sqrt[n]{(2^3)^n}$

$= \sqrt[n]{2} \cdot 2^3 \xrightarrow{n\to\infty} 8$

$\lim_{n\to\infty}\left(2^{\frac{1}{n}}\right) = y \rightarrow \ln y = \lim_{n\to\infty}\left(\ln\left(2^{\frac{1}{n}}\right)\right)$

$= \lim_{n\to\infty}\left(\frac{1}{n}\ln(2)\right)$

$\ln y = 0 \rightarrow$

$y = e^0 = 1$

47. $a_n = \left(1 + \dfrac{2}{n}\right)^n$ Same technique

$\left(1+\frac{2}{n}\right)^{\frac{n}{2}\cdot 2} = \left(\left(1+\frac{2}{n}\right)^{\frac{n}{2}}\right)^2 \xrightarrow{n\to\infty} e^2$ as previous

$\left(1+\frac{2}{n}\right)^{\frac{n}{2}} \xrightarrow{n\to\infty} e$

$\left(1+\frac{1}{x}\right)^{x} \xrightarrow{x\to\infty} e$

$(1+x)^{\frac{1}{x}} \xrightarrow{x\to 0} e$

$(1+x)^{\frac{1}{x}} \xrightarrow{x\to\infty} e$

check I ain't blowin' smoke

48. $a_n = \dfrac{\sin 2n}{1 + \sqrt{n}}$

Squeeze!

$-1 \le \sin(2n) \le 1$

$\dfrac{-1}{1+\sqrt{n}} \le \dfrac{\sin(2n)}{1+\sqrt{n}} \le \dfrac{1}{1+\sqrt{n}}$

$n \to \infty$: $\dfrac{-1}{1+\sqrt{n}} \to 0$, $\dfrac{1}{1+\sqrt{n}} \to 0$

$0 \le \lim_{n\to\infty} \dfrac{\sin(2n)}{1+\sqrt{n}} \le 0$

$\lim_{n\to\infty} \dfrac{\sin(2n)}{1+\sqrt{n}} = 0$