10.4 Areas and Lengths in Polar Coordinates

In this section we develop the formula for the area of a region whose boundary is given by a polar equation. We need to use the formula for the area of a sector of a circle:

1 $$\boxed{A = \frac{1}{2}r^2\theta}$$

πr^2 = Area of circle.

Area $\sim$ Angle

$\theta = 2\pi \Rightarrow \pi r^2 = (2\pi)\left(\frac{1}{2}r^2\right)$

proportionality constant.

is your θ.

where, as in Figure 1, r is the radius and θ is the radian measure of the central angle.

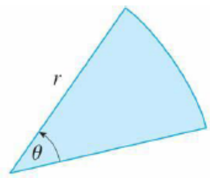


Figure 1

Formula 1 follows from the fact that the area of a sector is proportional to its central angle:

$$A = (\theta/2\pi)\pi r^2 = \tfrac{1}{2}r^2\theta.$$

Let $\mathcal{R}$ be the region, illustrated in Figure 2, bounded by the polar curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$, where f is a positive continuous function and where $0 < b - a \leq 2\pi$.

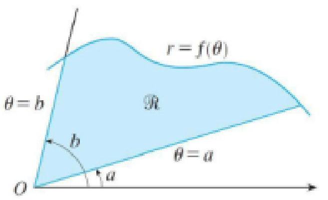


Figure 2

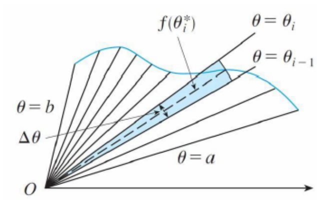


Figure 3

Again, the assumption that angles are equal

$\Delta A_i \approx \frac{1}{2}[f(\theta_i^*)]^2 \, \Delta\theta$

Any point in there.

$$\boxed{2} \quad A \approx \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i^*)]^2 \, \Delta\theta \xrightarrow{n \to \infty} \boxed{3} \quad \int_a^b \frac{1}{2}[f(\theta)]^2 \, d\theta \quad = \quad \boxed{4} \quad \int_a^b \frac{1}{2} r^2 \, d\theta$$

The rays $\theta = \theta_i$ then divide $\mathcal{R}$ into n smaller regions with central angle $\Delta\theta = \theta_i - \theta_{i-1}$. If we choose θ_i^* in the ith subinterval $[\theta_{i-1}, \theta_i]$, then the area ΔA_i of the ith region is approximated by the area of the sector of a circle with central angle $\Delta\theta$ and radius $f(\theta_i^*)$. (See Figure 3.)

So $n \to \infty$ guarantees $\Delta\theta_i \to 0$

We divide the interval $[a, b]$ into subintervals with endpoints $\theta_0, \theta_1, \theta_2, \ldots, \theta_n$ and equal width $\Delta\theta$.

Example

Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

All 3 kinds of symmetry

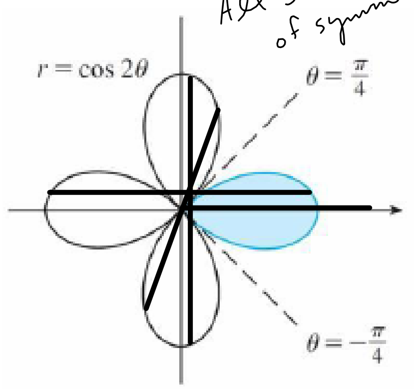


Figure 4

$$A = \int_{-\pi/4}^{\pi/4} \tfrac{1}{2} r^2\, d\theta = 2\int_0^{\frac{\pi}{4}} \tfrac{1}{2} r^2 d\theta$$

$$= \tfrac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta\, d\theta$$

$$= \int_0^{\pi/4} \cos^2 2\theta\, d\theta$$

$$= \int_0^{\pi/4} \tfrac{1}{2}(1 + \cos 4\theta)\, d\theta$$

$$= \tfrac{1}{2}\Big[\theta + \tfrac{1}{4}\sin 4\theta\Big]_0^{\pi/4}$$

$$= \frac{\pi}{8}$$

Arc Length

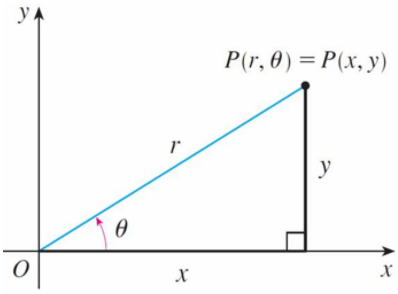


Figure 5

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

$$r = f(\theta)$$

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

Not exactly new. Below is from 10.3. Already been doin' this.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

This is basically the work done for #55 a.

$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\,\sin\theta + r^2\sin^2\theta$$

$$+ \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\,\cos\theta + r^2\cos^2\theta$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

So the 10.3 formula for arc length...

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

takes the following form in polar coordinates:

$$\boxed{5} \quad L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

S is what I keep using

Surface Area

#55 (a) $2\pi \int y\, ds$

$= 2\pi \int r\sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

S' in book for Surface Area.

Also 10.3. "Spirograph Pictures."

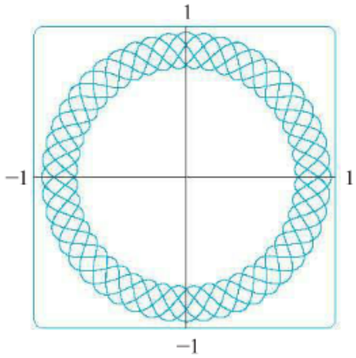


FIGURE 16
$r = \sin^2(2.4\theta) + \cos^4(2.4\theta)$

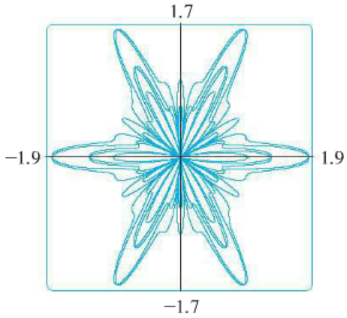
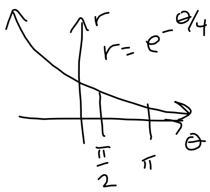
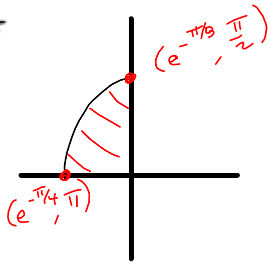


FIGURE 17
$r = \sin^2(1.2\theta) + \cos^3(6\theta)$

1–4 Find the area of the region that is bounded by the given curve and lies in the specified sector.

1. $r = e^{-\theta/4}, \quad \pi/2 \leqslant \theta \leqslant \pi$





$$\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \left(e^{-\theta/4}\right)^2 d\theta \qquad \left(e^{-\frac{\theta}{4}}\right)^2 = e^{-\frac{\theta}{4}\cdot 2} = e^{-\frac{\theta}{2}}$$

$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} e^{-\frac{\theta}{2}} d\theta$$

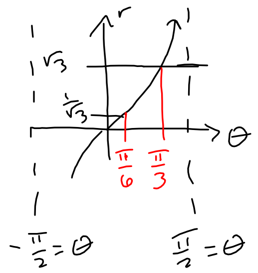
$$= \frac{1}{2}(-2)\int_{\frac{\pi}{2}}^{\pi} \left(e^{-\frac{\theta}{2}}\right)\left(-\tfrac{1}{2}d\theta\right)$$

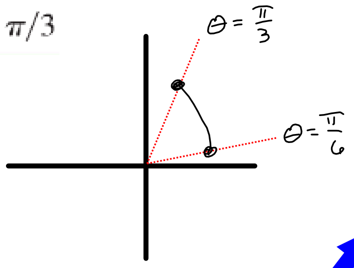
$$= -\left[e^{-\theta/2}\right]_{\frac{\pi}{2}}^{\pi}$$

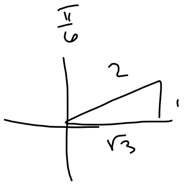
$$= -\left[e^{-\frac{\pi}{2}} - e^{-\frac{\pi}{4}}\right] = e^{-\frac{\pi}{4}} - e^{-\frac{\pi}{2}}$$

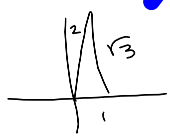
$$\int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}\cdot\left(e^{-\frac{\theta}{4}}\right)^2 d\theta = \left(e^{\frac{1}{4}\pi} - 1\right)e^{-\frac{1}{2}\pi} = \frac{1}{\left(e^{\pi}\right)^{1/4}} - \frac{1}{\sqrt{e^{\pi}}} \approx 0.2480585514$$

4. $r = \tan\theta, \quad \pi/6 \leqslant \theta \leqslant \pi/3$









$$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^2\theta \, d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\sec^2\theta - 1)\, d\theta$$

$$= \frac{1}{2}\Big[\tan\theta - \theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left[\sqrt{3} - \frac{\pi}{3} - \left(\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2}\left[\sqrt{3} - \frac{\pi}{3} - \frac{1}{\sqrt{3}} + \frac{\pi}{6}\right]$$

$$= \frac{1}{2}\left[\frac{6\cdot 3}{6\sqrt{3}} - \frac{2\sqrt{3}\pi}{6\sqrt{3}} - \frac{6}{6\sqrt{3}} + \frac{\sqrt{3}\pi}{6\sqrt{3}}\right]$$

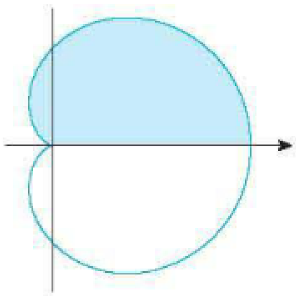
$$= \frac{18 - 2\sqrt{3}\pi - 6 + \sqrt{3}\pi}{12\sqrt{3}}$$

$$= \frac{12 - \sqrt{3}\pi}{12\sqrt{3}} \approx .3155508815$$

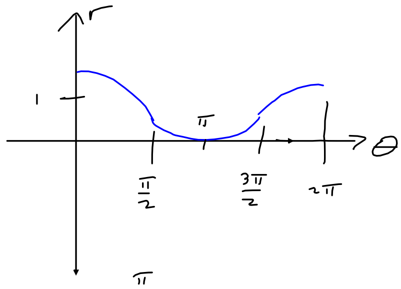
$$\int_{\frac{\mathrm{Pi}}{6}}^{\frac{\mathrm{Pi}}{3}} \frac{1}{2}\cdot(\tan(x))^2 \, dx = \frac{1}{3}\sqrt{3} - \frac{1}{12}\pi \approx 0.3155508815$$

5–8 Find the area of the shaded region.

6.



$r = 1 + \cos\theta$



$$(\cos\theta + 1)^2 = \cos^2\theta + 2\cos\theta + 1$$
$$= \frac{1}{2}(\cos(2\theta) + 1) + 2\cos\theta + 1$$

$$\frac{1}{2}\int_0^{\pi} \frac{1}{2}(\cos(2\theta) + 1)\,d\theta + \frac{1}{2}\int_0^{\pi} 2\cos\theta\,d\theta + \frac{1}{2}\int_0^{\pi} d\theta$$

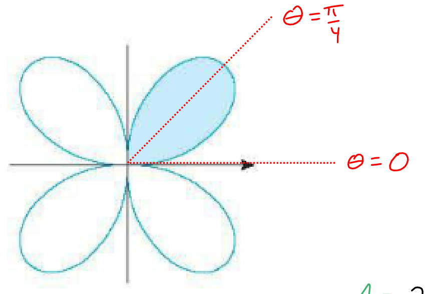
$$= \frac{1}{4}\int_0^{\pi} \cos(2\theta)\,d\theta + \frac{1}{4}\int_0^{\pi} d\theta$$
$$+ \int_0^{\pi} \cos\theta\,d\theta + \frac{1}{2}\int_0^{\pi} d\theta$$

$$= \frac{1}{4}\left(\frac{1}{2}\right)\int_0^{\pi} \cos(2\theta)\,2\,d\theta + \frac{3}{4}\int_0^{\pi} d\theta$$
$$+ \int_0^{\pi} \cos\theta\,d\theta$$

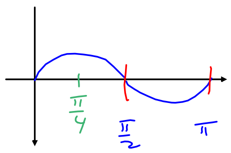
$$= \frac{1}{8}\Big[\sin(2\theta)\Big]_0^{\pi} + \frac{3}{4}\Big[\theta\Big]_0^{\pi} + \Big[\sin\theta\Big]_0^{\pi}$$

$$= \frac{1}{8}\Big[\sin(2\pi) - \sin(0)\Big]$$
$$+ \frac{3}{4}\pi - 0$$
$$+ \sin(\pi) - \sin(0)$$
$$= \frac{3}{4}\pi$$

8.



$r = \sin 2\theta$



$\sin(2\theta)$ π 2π $\sin\theta$

$$A = 2 \cdot \frac{1}{2} \int_0^{\frac{\pi}{4}} \sin^2(2\theta)\, d\theta$$

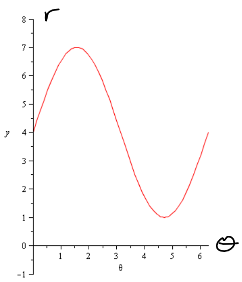
$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} (1 - \cos(4\theta)\, d\theta$$

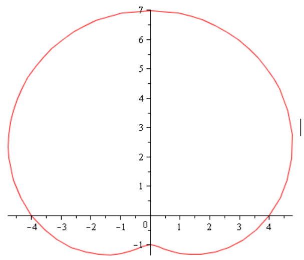
$$= \frac{1}{2} \left[\theta\right]_0^{\frac{\pi}{4}} + \frac{1}{2} \cdot \frac{1}{4} \int_0^{\frac{\pi}{4}} \cos(4\theta)\, 4\, d\theta$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - 0\right] + \left[\frac{1}{8} \sin(4\theta)\right]_0^{\frac{\pi}{4}} = \frac{1}{8} \sin \pi - \frac{1}{8} \sin(0) \boxed{= \frac{\pi}{8}}$$

9–12 Sketch the curve and find the area that it encloses.

12. $r = 4 + 3\sin\theta$





$$\frac{1}{2}\int_0^{2\pi}(4+3\sin\theta)^2\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(16+24\sin\theta+9\left(\tfrac{1}{2}\right)(1-\cos(2\theta))\right)d\theta$$

$$= 8\theta\Big]_0^{2\pi} - 12\cos\theta\Big]_0^{2\pi} + \frac{9}{4}\theta\Big]_0^{2\pi} - \frac{9}{4}\cdot\frac{1}{2}\sin(2\theta)\Big]_0^{2\pi}$$

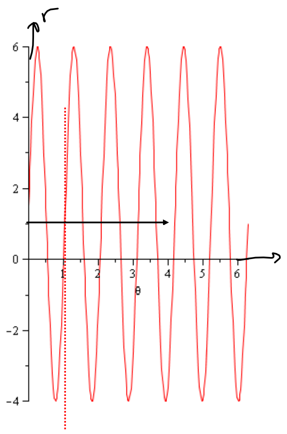
$$= 16\pi - 12\cos(2\pi) + 12\cos(0) + \frac{9\pi}{2} - 0 - \frac{9}{8}(\sin(4\pi) - \sin(0))$$

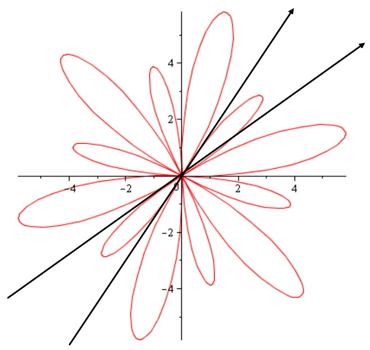
$$= 16\pi - 12 + 12 + \frac{9\pi}{2}$$

$$= \frac{(32+9)\pi}{2} = \frac{41\pi}{2}$$

13–16 Graph the curve and find the area that it encloses.

16. $r = 1 + 5 \sin 6\theta$

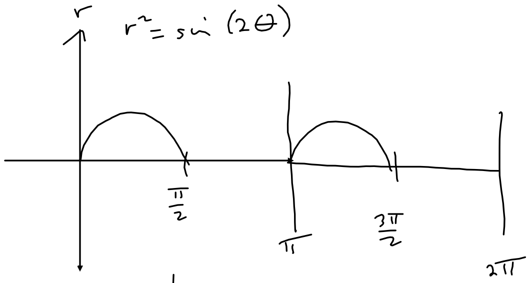




$$\int_0^{2 \cdot \mathrm{Pi}} \frac{1}{2} \cdot \left(1 + 5 \cdot \sin(6 \cdot \theta)\right)^2 \, d\theta = \frac{27}{2}\pi \approx 42.41150083$$

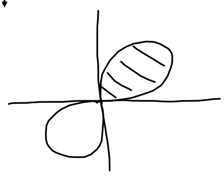
17–21 Find the area of the region enclosed by one loop of the curve.

18. $r^2 = \sin 2\theta$



$r = \pm\sqrt{\sin(2\theta)}$

$\frac{1}{2}\int_0^{\frac{\pi}{2}} r^2\, d\theta$



$= \frac{1}{2}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(2\theta)\,(2\,d\theta)$

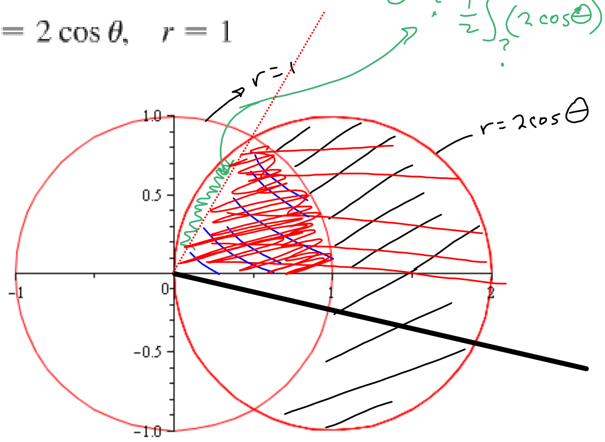
$= \frac{1}{4}\left(-\cos(2\theta)\right)\Big|_0^{\frac{\pi}{2}} = -\frac{1}{4}\left[-\cos(\pi) - (-\cos(0))\right]$

$= -\frac{1}{4}\left[-(-1) - (-1)\right] = \boxed{\frac{1}{2}}$

23–28 Find the area of the region that lies inside the first curve and outside the second curve.

23. $r = 2\cos\theta, \quad r = 1$

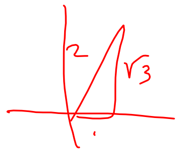
$\theta = ?$ $\frac{1}{2}\int_{?}^{\frac{\pi}{2}} (2\cos\theta)^2 \, d\theta$



$\equiv \frac{1}{2}\int_0^{?} (2\cos\theta)^2 - 1^2$

$2\cos\theta = 1 \implies ? = \frac{\pi}{3}$

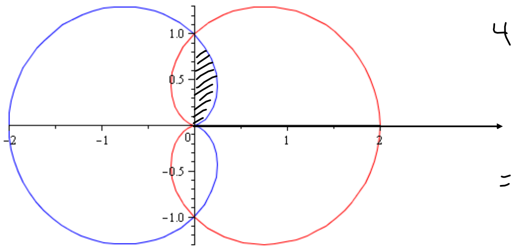
$\cos\theta = \frac{1}{2}$



$$\frac{1}{2}\int_0^{\frac{\pi}{3}} \left((2\cos(\theta))^2 - 1^2\right) d\theta + \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 4\cos^2\theta \, d\theta$$

29–34 Find the area of the region that lies inside both curves.

30. $r = 1 + \cos\theta, \quad r = 1 - \cos\theta$



$$4\int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos\theta)^2\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

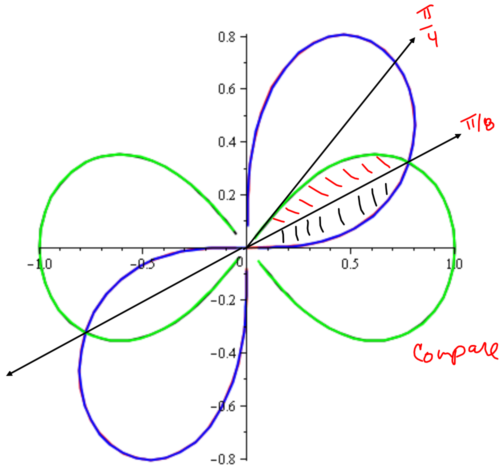
$$= 2\left[\theta - 2\sin\theta\right]_0^{\frac{\pi}{2}} + 2\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} (1 + \cos(2\theta))\, d\theta$$

$$= 2\cdot\frac{\pi}{2} - 4\sin\frac{\pi}{2} + \int_0^{\frac{\pi}{2}} d\theta + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2\theta)\, 2d\theta$$

$$= \pi - 4 + \frac{\pi}{2} + \left[\sin(2\theta)\right]_0^{\frac{\pi}{2}}$$

$$\boxed{= \frac{3\pi}{2} - 4}$$

33. $r^2 = \sin 2\theta, \quad r^2 = \cos 2\theta$



$\cos(2\theta) = \sin(2\theta)$

$\cos(2\theta) - \sin(2\theta) = 0$

$2\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

$\theta = \frac{\pi}{8}, \frac{3\pi}{8}$

Compare

$$4\int_0^{\frac{\pi}{8}} \frac{1}{2}\sin(2\theta)\,d\theta = \int_0^{\frac{\pi}{8}} \sin(2\theta)(2d\theta)$$

$$= \Big[-\cos(2\theta)\Big]_0^{\frac{\pi}{8}} = -\cos\frac{\pi}{4} - (-\cos(0))$$

$$= -\frac{1}{\sqrt{2}} + 1 = \frac{\sqrt{2}-1}{\sqrt{2}} \approx .292893$$

to

$$2\int_0^{\frac{\pi}{8}} \frac{1}{2}\sin(2\theta)\,d\theta + 2\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{1}{2}\cos(2\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{8}} \sin(2\theta)(2d\theta) + \frac{1}{2}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \cos(2\theta)(2d\theta)$$

$$= \frac{1}{2}\Big[-\cos(2\theta)\Big]_0^{\frac{\pi}{8}} + \frac{1}{2}\Big[\sin(2\theta)\Big]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\Big[-\cos\left(\frac{\pi}{4}\right) - (-\cos(0))\Big] + \frac{1}{2}\Big[\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{4}\right)\Big]$$

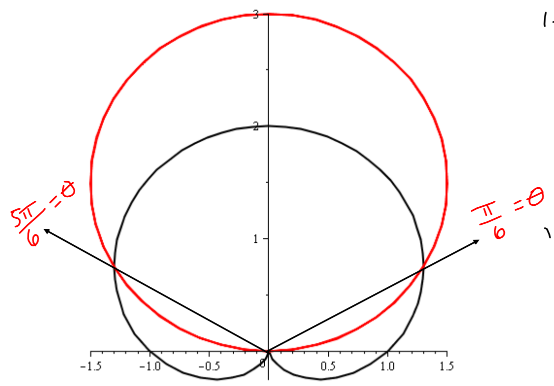
$$= \frac{1}{2}\Big[-\frac{1}{\sqrt{2}} + 1 + 1 - \frac{1}{\sqrt{2}}\Big]$$

$$= \frac{1}{2}\Big[2 - \frac{2}{\sqrt{2}}\Big] = 1 - \frac{1}{\sqrt{2}} \checkmark$$

37–42 Find all points of intersection of the given curves.

37. $r = 1 + \sin\theta, \quad r = 3\sin\theta$

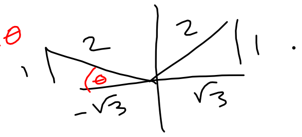
$1 + \sin\theta = 3\sin\theta$



$1 - 2\sin\theta = 0$

$-2\sin\theta = -1$

$\sin\theta = \frac{1}{2}$



$\frac{\pi}{6}, \frac{5\pi}{6}$

$r = 1 + \sin\frac{\pi}{6}$

$= 1 + \frac{1}{2} = \frac{3}{2}$

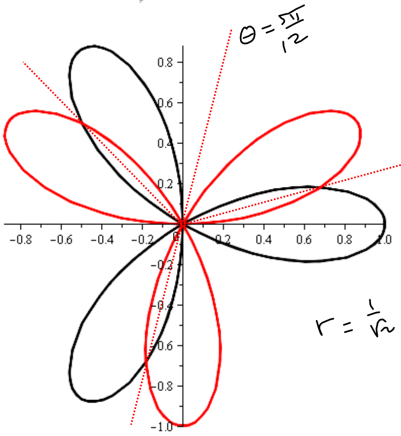
$(r, \theta) = (\frac{3}{2}, \frac{\pi}{6})$

or

$(\frac{3}{2}, \frac{5\pi}{6})$

Also share the pole.

40. $r = \cos 3\theta, \quad r = \sin 3\theta$



$\tan(3\theta) = 1$

$3\theta = \frac{\pi}{4}$

$\cos 3\theta = \sin 3\theta$

$3\theta = \frac{\pi}{4} + n\pi \quad , n \in \mathbb{Z}, \text{so}$

$\theta = \frac{\pi}{12} + n\frac{\pi}{3} \quad , n \in \mathbb{Z}$

Same @ pole.

$\frac{\pi}{12}, \quad \frac{\pi}{12} + \frac{4\pi}{12} = \frac{5\pi}{12},$

$\frac{\pi}{12} + \frac{2\pi}{3} = \frac{\pi}{12} + \frac{8\pi}{12} = \frac{9\pi}{12} = \frac{3\pi}{4}$

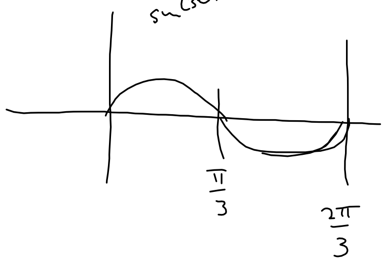
$r = \frac{1}{\sqrt{2}}$ @ $\theta = \frac{\pi}{12}$

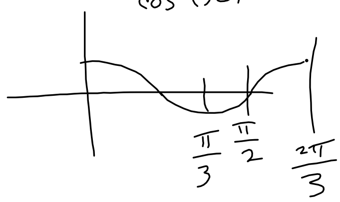
$\boxed{\left(\frac{1}{\sqrt{2}}, \frac{\pi}{12}\right), \left(-\frac{1}{\sqrt{2}}, \frac{5\pi}{12}\right), \left(\frac{1}{\sqrt{2}}, \frac{3\pi}{4}\right) \text{ plus @ the pole}}$

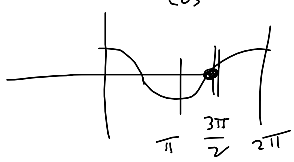
$\frac{1}{\sqrt{2}}$

$\frac{\frac{\pi}{3} + \frac{2\pi}{3}}{2} = \frac{3\pi/3}{2} = \frac{\pi}{2}$

$\cos\left(3 \cdot \frac{5\pi}{12}\right) = \cos\left(\frac{5\pi}{4}\right)$







45–48 Find the exact length of the polar curve.

45. $r = 2\cos\theta, \quad 0 \leqslant \theta \leqslant \pi$

$$L = \int_0^{\pi} ds = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\pi} \sqrt{(4\cos^2\theta + 4\sin^2\theta)}\, d\theta = \int_0^{\pi} 2\, d\theta = 2\pi$$

47. $r = \theta^2, \quad 0 \leq \theta \leq 2\pi$

$r^2 = \theta^4, \left(\frac{dr}{d\theta}\right)^2 = (2\theta)^2 = 4\theta^2$

$\int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta = \int_0^{2\pi} \theta\sqrt{\theta^2+4}\, d\theta$

$u = \theta^2 + 4$

$= \frac{1}{2}\int_0^{2\pi} \sqrt{\theta^2+4}\,(2\theta\, d\theta)$

$= \frac{1}{2}\frac{(\theta^2+4)^{\frac{3}{2}}}{\frac{3}{2}}\Big]_0^{2\pi} = \frac{1}{2}\cdot\frac{2}{3}(\theta^2+4)^{\frac{3}{2}}\Big]_0^{2\pi} = \frac{1}{3}\left[(4\pi^2+4)^{\frac{3}{2}} - 4^{\frac{3}{2}}\right]$

$= \boxed{\frac{1}{3}(4\pi^2+4)^{\frac{3}{2}} - 8}$ OK

$= \frac{1}{3}\left(\left(4(\pi^2+1)\right)^{\frac{3}{2}} - 8\right)$

$= \frac{1}{3}\left(8(\pi^2+1)^{\frac{3}{2}} - 8\right)$

$= \frac{8}{3}(\pi^2+1)^{3/2} - \frac{8}{3}$

$\left(4(\pi^2+1)\right)^{\frac{3}{2}}$

$= 4^{\frac{3}{2}}(\pi^2+1)^{\frac{3}{2}}$