

Polar Coordinates

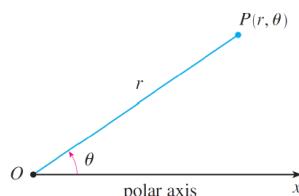
Picture1

A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. Usually we use Cartesian coordinates, which are directed distances from two perpendicular axes.

Here we describe a coordinate system introduced by Newton, called the **polar coordinate system**, which is more convenient for many purposes.

We choose a point in the plane that is called the **pole** (or origin) and is labeled O . Then we draw a ray (half-line) starting at O called the **polar axis**.

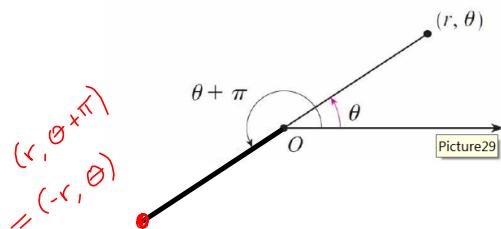
This axis is usually drawn horizontally to the right and corresponds to the positive x -axis in Cartesian coordinates. If P is any other point in the plane, let r be the distance from O to P and let θ be the angle (usually measured in radians) between the polar axis and the line OP as in Figure 1.



We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction.

If $P = O$, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of θ .

We extend the meaning of polar coordinates (r, θ) to the case in which r is negative by agreeing that, as in Figure 2, the points $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance $|r|$ from O , but on opposite sides of O .



If $r > 0$, the point (r, θ) lies in the same quadrant as θ ; if $r < 0$, it lies in the quadrant on the opposite side of the pole. Notice that $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.

Polar Coordinates

The connection between polar and Cartesian coordinates can be seen from Figure 5, in which the pole corresponds to the origin and the polar axis coincides with the positive x-axis.

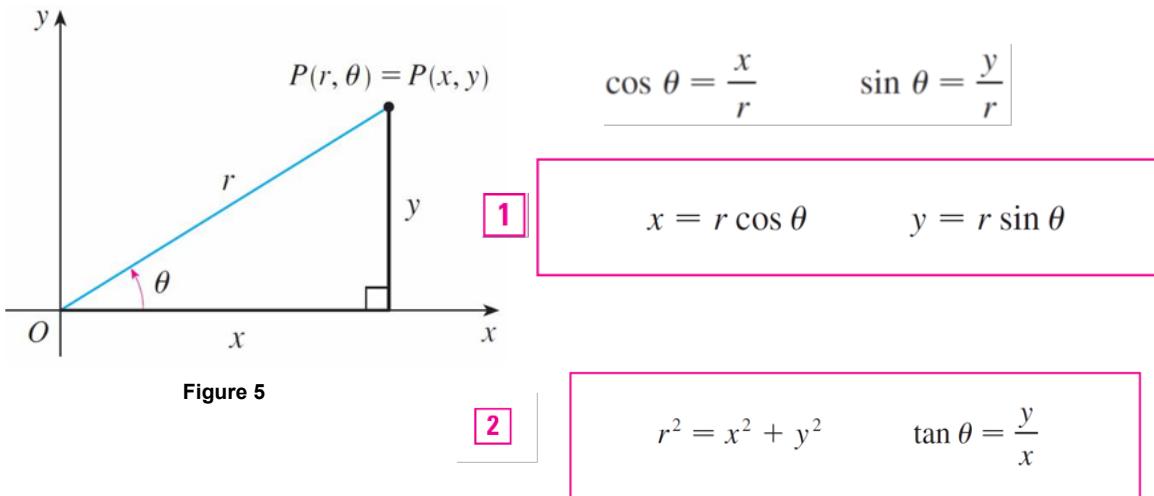


Figure 5

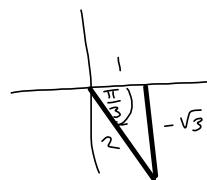
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Equations 2 do not uniquely determine r, θ . Make sure you know what quadrant (x, y) is in.

$$(x, y) = (1, -\sqrt{3})$$

This one's ok...

$$\tan \theta = -\frac{\sqrt{3}}{1} = -\sqrt{3}$$



$$\arctan(\tan \theta) = \arctan(-\sqrt{3}) = -\frac{\pi}{3}$$

$$r^2 = x^2 + y^2 = 1^2 + (-\sqrt{3})^2 = 1+3=4$$

$$\Rightarrow r = \pm\sqrt{4} = \pm 2$$

$$r = +2$$

$$(r, \theta) = (2, -\frac{\pi}{3}) = (2, \frac{5\pi}{3})$$

$$2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

But this one?

$$(x, y) = (-1, -\sqrt{3})$$

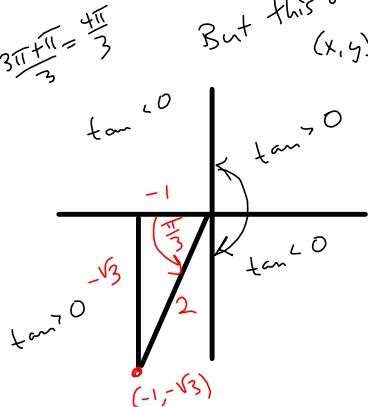
There's where [2] falls short.

$$\tan \theta = \frac{-\sqrt{3}}{-1} = \sqrt{3}$$

$$\tan^{-1}(\tan \theta) - \arctan(\sqrt{3}) = 60^\circ = \frac{\pi}{3}$$

$$r = 2 \Rightarrow (2, \frac{\pi}{3}) = (r, \theta)$$

$$\neq (2, \frac{4\pi}{3}) \longleftrightarrow (-1, -\sqrt{3})$$



The **graph of a polar equation** $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

Might have

won't all be...

$$r = 2 + 3 \cos \theta$$

$$\sqrt{(2+3 \cos \theta)^2} \sin \theta + 7 = -\cos \theta r$$
$$\sqrt{(2+3 \cos \theta)^2} \sin \theta + 7 + r \cos \theta = 0$$

$\underbrace{\hspace{100pt}}$

$F(r, \theta)$

Example 4

What curve is represented by the polar equation $r = 2$?

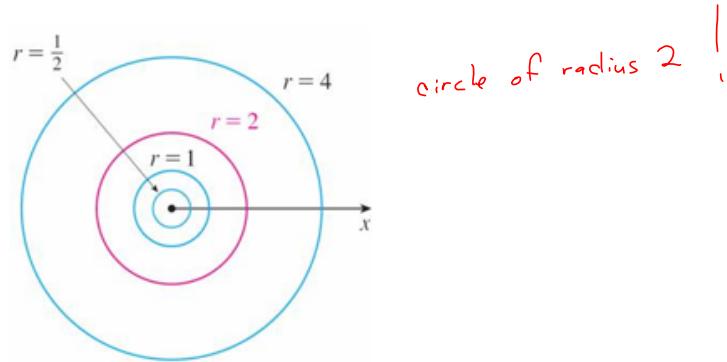
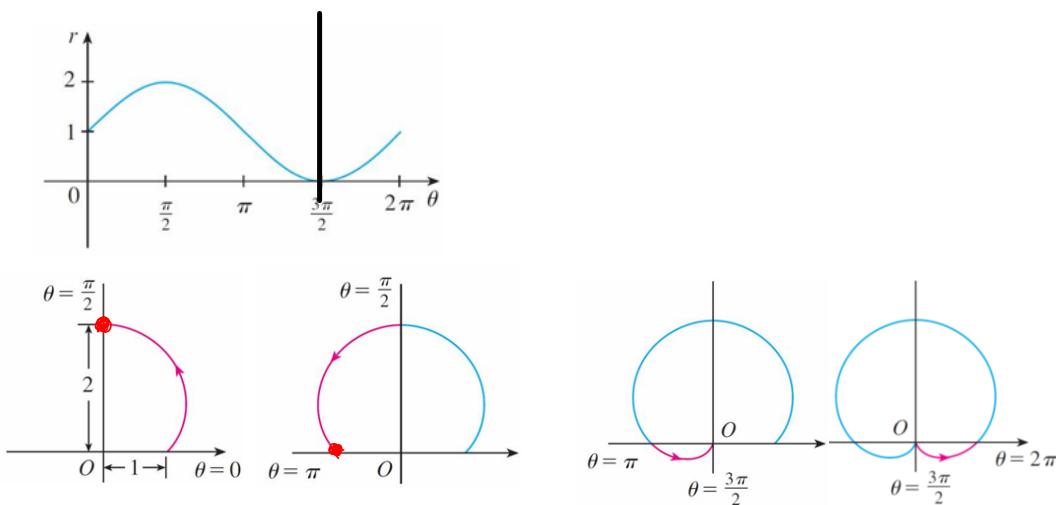


Figure 6

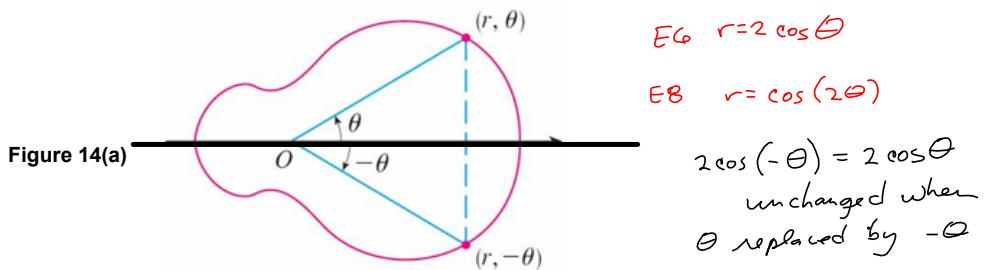
Example 7

Sketch the curve $r = 1 + \sin \theta$.



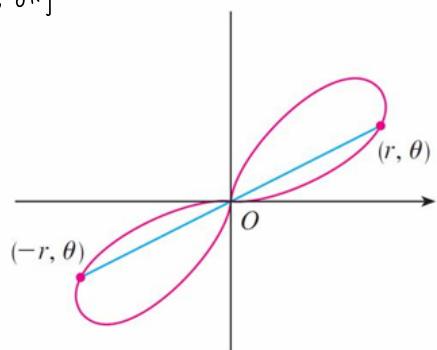
Symmetry

- (a) If a polar equation is unchanged when θ is replaced by $-\theta$, the curve is symmetric about the polar axis. \rightarrow x-axis



- (b) If the equation is unchanged when r is replaced by $-r$, or when θ is replaced by $\theta + \pi$, the curve is symmetric about the pole. (This means that the curve remains unchanged if we rotate it through 180° about the origin.)

thru the origin



$$r^2 = \cos \theta$$

$$(-r)^2 = r^2 = \cos \theta$$

$$r = \cos(2\theta)$$

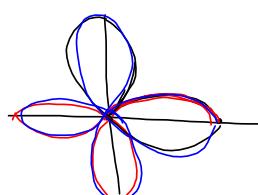
$$-r = \cos(2\theta) \quad \text{Check No}$$

OR

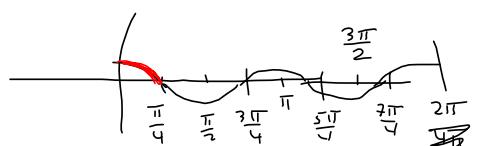
$$r = \cos(2(\theta + \pi))$$

$$r = \cos(2\theta + 2\pi)$$

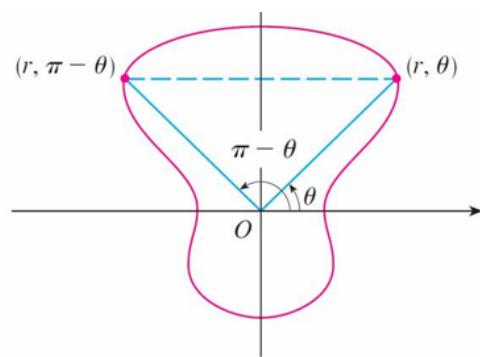
$$r = \cos(2\theta)$$



$$\cos(2\theta)$$



- (c) If the equation is unchanged when θ is replaced by $\pi - \theta$, the curve is symmetric about the vertical line $\theta = \pi/2$.



$$\begin{aligned}
 & \text{if } r = 1 + \sin \theta \\
 & r = 1 + \sin(\pi - \theta) ? \\
 & = 1 + \sin(\pi)\cos(-\theta) + \sin(-\theta)\cos(\pi) \\
 & = 1 + 0 - (\sin \theta)(-1) = 1 + \sin \theta \\
 & \text{is the same!}
 \end{aligned}$$

Now, assume $r = f(\theta)$ is a function, too!

$$x = r \cos \theta, y = r \sin \theta \text{ & } \text{S10.2 say}$$

$$\begin{aligned} x &= r \cos \theta \\ f &= r, g = \cos \theta \\ (fg)' &= f'g + fg' \\ &= \frac{dr}{d\theta} \cos \theta + r(-\sin \theta) \end{aligned}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}$$

$$\text{Horizontal: } \frac{dy}{d\theta} = 0$$

In case of tie,

$$\text{Vertical: } \frac{dx}{d\theta} = 0$$

use L'HOPITAL'S.
Could be vert/horiz/neither.

Example 9

(a) For the cardioid $r = 1 + \sin \theta$, find the slope of the tangent line when $\theta = \pi/3$.

(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

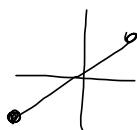
Using Equation 3 with $r = 1 + \sin \theta$, we have

Symmetry: polar axis:

$$\begin{aligned} r &= 1 + \sin(-\theta) \\ &= 1 - \sin \theta \neq \text{same} \end{aligned}$$

$$\theta = \frac{\pi}{2}$$

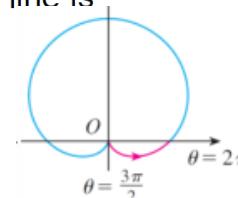
$$r = 1 + \sin(\pi - \theta)$$



pole: $-r = 1 + \sin \theta$

$$\begin{aligned} r &= -1 - \sin \theta \neq \text{same} \\ r &= 1 + \sin(\pi + \theta) \\ &= 1 + \sin \pi \cos \theta + \sin \theta \cos \pi \\ &= 1 - \sin \theta \neq \text{same} \end{aligned}$$

$$r = 1 + \sin(\theta) = \text{same!}$$

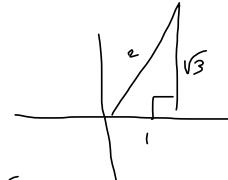


Symmetry about $\theta = \frac{\pi}{2}$
we can graph from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$
& then reflect!

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} = \frac{\cos \theta \sin \theta + (1 + \sin \theta) \cos \theta}{\cos \theta \cos \theta - (1 + \sin \theta) \sin \theta} \end{aligned}$$

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$$= \frac{\cos \theta (1 + 2 \sin \theta)}{1 - 2 \sin^2 \theta - \sin \theta} = \frac{\cos \theta (1 + 2 \sin \theta)}{(1 + \sin \theta)(1 - 2 \sin \theta)}$$



$$\begin{aligned} (\text{a}) \quad \frac{dy}{dx} \Big|_{\theta=\frac{\pi}{3}} &= \frac{(\cos \frac{\pi}{3})(1 + 2 \sin \frac{\pi}{3})}{(1 + \sin \frac{\pi}{3})(1 - 2 \sin \frac{\pi}{3})} = \frac{(\frac{1}{2})(1 + 2(\frac{\sqrt{3}}{2}))}{(1 + \frac{\sqrt{3}}{2})(1 - 2(\frac{\sqrt{3}}{2}))} \\ &= \frac{\frac{1}{2}(1 + \sqrt{3})}{(\frac{1 + \sqrt{3}}{2})(1 - \sqrt{3})} = \frac{1 + \sqrt{3}}{(1 + \sqrt{3})(1 - \sqrt{3})} = \frac{1 + \sqrt{3}}{2 - \sqrt{3} - 3} = \frac{1 + \sqrt{3}}{-1 - \sqrt{3}} = -1 \end{aligned}$$

Eq'n of tan line in rect. coords.

$$(x_1, y_1) = \left(\frac{2+\sqrt{3}}{4}, \frac{2\sqrt{3}+3}{4} \right)$$

$$\begin{aligned} y &= -1(x - \left(\frac{2+\sqrt{3}}{4} \right)) + \frac{2\sqrt{3}+3}{4} \\ y &= m(x - x_1) + y_1 \end{aligned}$$

$$\begin{aligned} x &= r \cos \theta \\ &= (1 + \sin \theta) \cos \theta \\ &= \left(1 + \frac{\sqrt{3}}{2}\right) \left(\frac{1}{2}\right) = \frac{1}{2} + \frac{\sqrt{3}}{4} \\ x \Big|_{\theta=\frac{\pi}{3}} &= \left(1 + \frac{\sqrt{3}}{2}\right) \left(\frac{\sqrt{3}}{2}\right) \\ y \Big|_{\theta=\frac{\pi}{3}} &= \left(1 + \frac{\sqrt{3}}{2}\right) \left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{\sqrt{3}}{2} + \frac{3}{4} \end{aligned}$$

Example 9 – Solution

cont'd

(a) The slope of the tangent at the point where $\theta = \pi/3$ is

$$\begin{aligned}\frac{dy}{dx} \Big|_{\theta=\pi/3} &= \frac{\cos(\pi/3)(1 + 2 \sin(\pi/3))}{(1 + \sin(\pi/3))(1 - 2 \sin(\pi/3))} \\ &= \frac{\frac{1}{2}(1 + \sqrt{3})}{(1 + \sqrt{3}/2)(1 - \sqrt{3})} = \frac{1 + \sqrt{3}}{(2 + \sqrt{3})(1 - \sqrt{3})} = \frac{1 + \sqrt{3}}{-1 - \sqrt{3}} = -1\end{aligned}$$

Example 9 – Solution

cont'd

(b) Observe that

$$\frac{dy}{d\theta} = \cos \theta (1 + 2 \sin \theta) = 0 \quad \text{when } \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\frac{dx}{d\theta} = (1 + \sin \theta)(1 - 2 \sin \theta) = 0 \text{ when } \theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

$$(2, \pi/2), \left(\frac{1}{2}, 7\pi/6\right), \left(\frac{1}{2}, 11\pi/6\right)$$

$\left(\frac{3}{2}, \pi/6\right)$ and $\left(\frac{3}{2}, 5\pi/6\right)$.

$$\begin{aligned} \lim_{\theta \rightarrow (3\pi/2)^-} \frac{dy}{dx} &= \left(\lim_{\theta \rightarrow (3\pi/2)^-} \frac{1 + 2 \sin \theta}{1 - 2 \sin \theta} \right) \left(\lim_{\theta \rightarrow (3\pi/2)^-} \frac{\cos \theta}{1 + \sin \theta} \right) \\ &= -\frac{1}{3} \lim_{\theta \rightarrow (3\pi/2)^-} \frac{\cos \theta}{1 + \sin \theta} = \frac{0}{0} \\ &\stackrel{\text{L'H}}{=} -\frac{1}{3} \lim_{\theta \rightarrow (3\pi/2)^-} \frac{-\sin \theta}{\cos \theta} = -\frac{1}{3} \cdot -\infty = \infty \end{aligned}$$

By symmetry,

$$\lim_{\theta \rightarrow (3\pi/2)^+} \frac{dy}{dx} = -\infty$$

$\cos \theta (1 + 2 \sin \theta) = 0$
 $\cos \theta = 0 \quad 1 + 2 \sin \theta = 0$
 $2 \sin \theta = -1 \quad \sin \theta = -\frac{1}{2}$

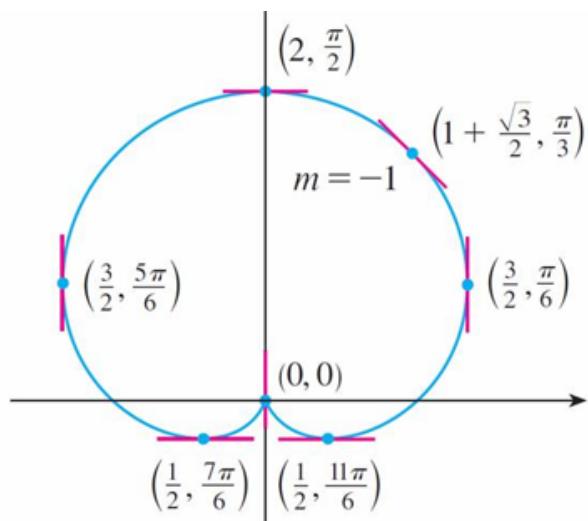
$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$
 $\theta = \frac{\pi}{6}, \frac{7\pi}{6}$
 $\theta = \frac{11\pi}{6}$

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Example 9 – Solution

cont'd

Thus there is a vertical tangent line at the pole
(see Figure 15).



Tangents to Polar Curves

Note:

Instead of having to remember Equation 3, we could employ the method used to derive it. For instance, in Example 9 we could have written

$$\begin{aligned}x &= r \cos \theta = (1 + \sin \theta) \cos \theta = \cos \theta + \frac{1}{2} \sin 2\theta \\y &= r \sin \theta = (1 + \sin \theta) \sin \theta = \sin \theta + \sin^2 \theta\end{aligned}$$

Then we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos \theta + 2 \sin \theta \cos \theta}{-\sin \theta + \cos 2\theta} = \frac{\cos \theta + \sin 2\theta}{-\sin \theta + \cos 2\theta}$$

Graphing Polar Curves with Graphing Devices

Although it's useful to be able to sketch simple polar curves by hand, we need to use a graphing calculator or computer when we are faced with a curve as complicated as the ones shown in Figures 16 and 17.

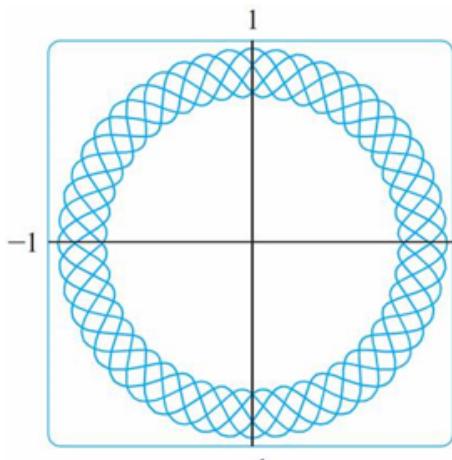


Figure 16

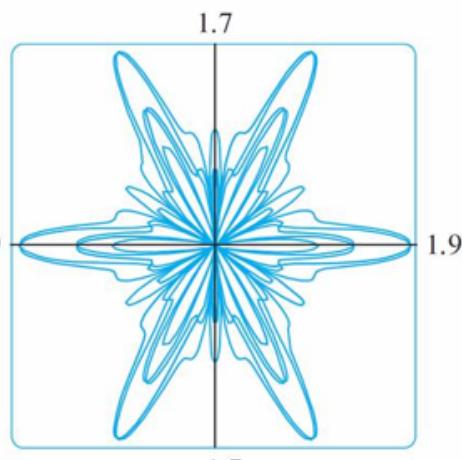
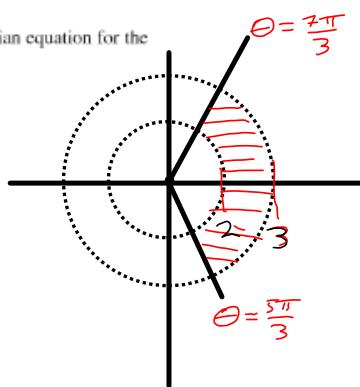


Figure 17

15-20 Identify the curve by finding a Cartesian equation for the curve.

11. $2 < r < 3, \quad 5\pi/3 \leq \theta \leq 7\pi/3$

$$2 < r < 3$$



$$17. r = 2 \cos \theta$$

$$r^2 = 2r \cos \theta$$

$$x^2 + y^2 = 2x$$

$$x^2 - 2x + y^2 = 0$$

$$x^2 - 2x + 1^2 + y^2 = 1$$

$$(x-1)^2 + y^2 = 1$$

Circle! $(h, k) = (1, 0)$
 $r = 1$

21-26 Find a polar equation for the curve represented by the given Cartesian equation.

| 25. $x^2 + y^2 = 2cx$

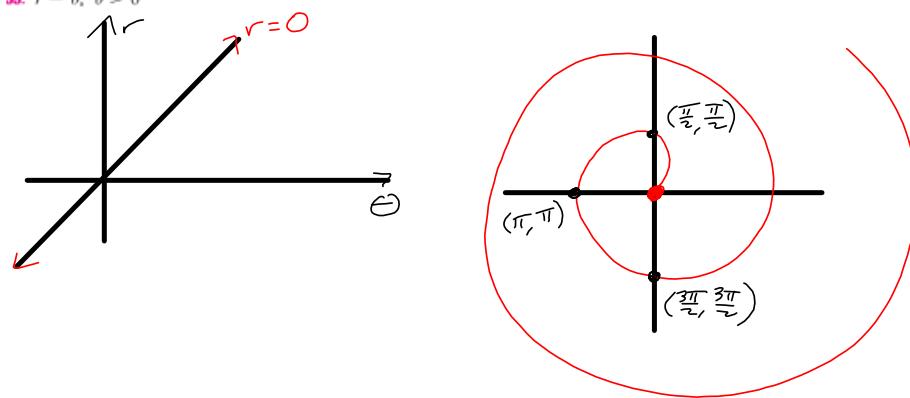
$$r^2 = 2cr\cos\theta$$

Symmetric thru
the pole

$$(-r)^2 = r^2 = 2cr\cos\theta$$

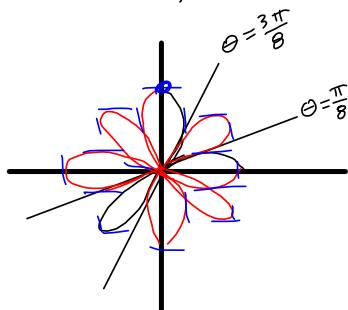
29–46 Sketch the curve with the given polar equation by first sketching the graph of r as a function of θ in Cartesian coordinates.

33. $r = \theta, \theta \geq 0$



37. $r = 2 \cos 4\theta$

All 3
Kinds of
symmetry:
Get it + $\frac{\pi}{2}$
from 0 to $\frac{\pi}{2}$.



Symmetry:

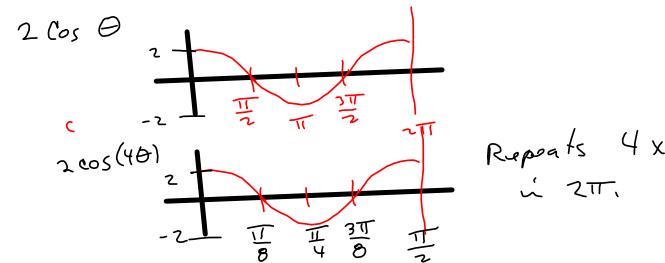
polar axis: Yes
 $r = 2 \cos(4(-\theta))$
 $= 2 \cos(-4\theta)$
 $= 2 \cos(4\theta)$

Pole: Yes
 $-r = 2 \cos(4\theta)$ No

$$\begin{aligned} r &= 2 \cos(4(\theta + \pi)) \\ &= 2 \cos(4\theta + 4\pi) \\ &= 2 [\cos(4\theta) \cos(4\pi) - \sin(4\theta) \sin(4\pi)] \\ &= 2 \cos(4\theta) \quad \text{Yes!} \end{aligned}$$

$$\theta = \frac{\pi}{2}: \quad r = 2 \cos(4(\pi - \theta))$$

$$\begin{aligned} &= 2 \cos(4\pi - 4\theta) \\ &= 2 [\cos(4\pi) \cos(-4\theta) - \sin(4\pi) \sin(-4\theta)] \\ &= 2 [1 \cdot \cos(4\theta) - 0 \cdot (-\sin(4\theta))] \\ &= 2 \cos(4\theta) \quad \text{Yes!} \end{aligned}$$



$$y = r \sin \theta = 2 \cos(4\theta) \sin \theta$$

$$\begin{aligned} \Rightarrow \frac{dy}{d\theta} &= 2 [-4 \sin(4\theta) \sin \theta + \cos(4\theta) \cos \theta] \\ &= -8 \sin(4\theta) \sin \theta + 2 \cos(4\theta) \cos \theta \quad \text{SET=0 is } \textcircled{1} \end{aligned}$$

$$x = r \cos \theta = 2 \cos(4\theta) \cos \theta$$

$$\begin{aligned} \Rightarrow \frac{dx}{d\theta} &= 2 [-4 \sin(4\theta) \cos \theta + (2 \cos(4\theta))(-\sin \theta)] \\ &= -8 \sin(4\theta) \cos \theta - 2 \cos(4\theta) \sin \theta \quad \text{SET=0 is } \textcircled{2} \end{aligned}$$

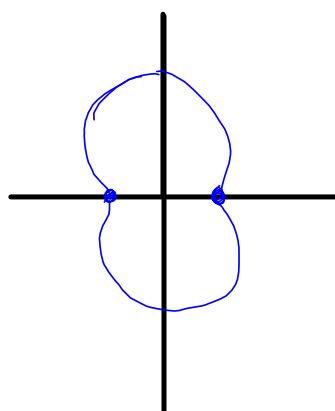
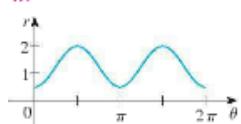
$$\textcircled{1} \quad -8 \sin(4\theta) \sin \theta = -2 \cos(4\theta) \cos \theta \quad \textcircled{2}$$

$$\frac{4 \sin 4\theta}{\cos 4\theta} = \frac{\cos \theta}{\sin \theta}$$

$$4 \tan(4\theta) = \cot \theta \quad \text{ugh!}$$

- 47–48 The figure shows a graph of r as a function of θ in Cartesian coordinates. Use it to sketch the corresponding polar curve.

47.

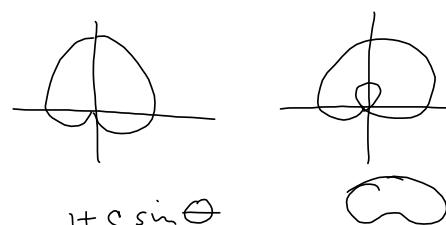
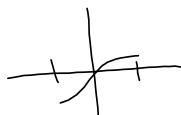


53. (a) In Example 11 the graphs suggest that the limaçon $r = 1 + c \sin \theta$ has an inner loop when $|c| > 1$. Prove that this is true, and find the values of θ that correspond to the inner loop.
 (b) From Figure 19 it appears that the limaçon loses its dimple when $c = \frac{1}{2}$. Prove this.

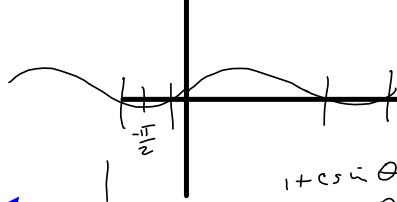
$$(a) |c| < 1 \Rightarrow 1 + c \sin \theta > 0$$

$$r = 1 + c \sin \theta \stackrel{\text{SET}}{=} 0$$

(b) Take out the dimple.



$$1 + c \sin \theta = 0$$



$$1 + c \sin \theta = 0$$

$$\sin \theta = -1$$

$$\sin \theta = -\frac{1}{c}$$

$$\sin^{-1}(\sin \theta) = \sin^{-1}(-\frac{1}{c})$$

$$\theta = \sin^{-1}(-\frac{1}{c})$$

55–60 Find the slope of the tangent line to the given polar curve at the point specified by the value of θ .

57. $r = 1/\theta, \theta = \pi$

$$x = r \cos \pi = \frac{1}{\pi} \cos \pi = \frac{1}{\pi}$$

$$r = \frac{1}{\theta} \quad \frac{dr}{d\theta} = -\frac{1}{\theta^2}$$

$$y = r \sin \pi = 0$$

$$(x, y) = \left(\frac{1}{\pi}, 0\right)$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta} \left[\frac{1}{\theta} \sin \theta \right] = -\frac{1}{\theta^2} \sin \theta + \frac{1}{\theta} \cos \theta$$

$$\begin{cases} y = -\pi(x - \frac{1}{\pi}) + 0 \\ y = -\pi x + 1 \end{cases}$$

$$\frac{dx}{d\theta} = \frac{d}{d\theta} \left[\frac{1}{\theta} \cos \theta \right] = -\frac{1}{\theta^2} \cos \theta - \frac{1}{\theta} \sin \theta$$

$$\left. \frac{dy}{dx} \right|_{\theta=\pi} = \frac{-\frac{1}{\pi^2} \sin \pi + \frac{1}{\pi} \cos \pi}{-\frac{1}{\pi^2} \cos \pi - \frac{1}{\pi} \sin \pi} = \frac{\frac{1}{\pi} \cos \pi}{\frac{1}{\pi^2}} = \frac{\cos \pi}{\pi} = -\frac{1}{\pi}$$

$$\boxed{-\pi = m}$$

61–64 Find the points on the given curve where the tangent line is horizontal or vertical.

61. $r = 3 \cos \theta$